Feedback Saddle Point Equilibria for Soft-Constrained Zero-Sum Linear Quadratic Descriptor Differential Game (penulis pertama)Feedback Saddle Point Equilibria for Soft-Constrained Zero-Sum Linear Quadratic Descriptor Differential Game

Authors:

Muhammad Wakhid Musthofa

Salmah

Ari Suparwanto

Jacob C Engwerda

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Feedback saddle point equilibria for soft-constrained zero-sum linear quadratic descriptor differential game

MUHAMMAD WAKHID MUSTHOFA, SALMAH, JACOB C. ENGWERDA and ARI SUPARWANTO

In this paper the feedback saddle point equilibria of soft-constrained zero-sum linear quadratic differential games for descriptor systems that have index one will be studied for a finite and infinite planning horizon. Both necessary and sufficient conditions for the existence of a feedback saddle point equilibrium are considered.

Key words: soft-constrained zero-sum linear quadratic differential game, feedback information structure, descriptor systems

1. Introduction

In the last decade significant progress has been made in the study of linear quadratic differential games. A linear quadratic differential game is a mathematical model that represents a conflict between different agents which control a dynamical system and each of them is trying to minimize his individual quadratic objective function by giving a control to the system. For this purpose, linear quadratic differential games have been applied in many different fields like economic and management science [25], [4], to study issues like e.g. economic competitions among companies and environmental management games; military studies, to study armed conflicts; or parlor games, see e.g. [11]. Moreover, in optimal control theory it is well known that the issue to obtain robust control strategies to solve the disturbance attenuation problem can be approached as a dynamic game problem [1, 16, 3, 20].

Although the theory has been applied in many fields, however, an extension of this theory is called for systems that can be modeled as a set of coupled differential and algebraic equations. These, so-called, descriptor systems can be used to model more accurately the structure of physical systems, including non-dynamic modes and impulsive modes [13]. Applications of descriptor systems can be found in chemical

M.W. Musthofa, the corresponding author, is Ph.D student of Mathematics Department Universitas Gadjah Mada, Indonesia; UIN Sunan Kalijaga, Yogyakarta Indonesia; email: mwakhid_m@yahoo.com. Salmah and A. Suparwanto are with Mathematics Department Universitas Gadjah Mada, Yogyakarta Indonesia. J.C. Engwerda is with Tilburg University, Tilburg, The Netherlands.

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processes [15], circuit systems [23, 24], economic systems [17], large-scale interconnected systems [18, 29], mechanical engineering systems [12], power systems [28], and robotics [19].

The study of differential games for descriptor systems was initiated by [10]. They investigated the well-posedness of closed-loop Nash strategies with respect to singular perturbations and presented a hierarchical reduction procedure to reduce the game to an ordinary game which is well-posed. Further references that studied the time-continuous games are [30, 31], while the discrete-time version of such games was studied by [32]. The index one case[1] was studied by [6] for an open-loop information structure and in [7, 8] for the feedback information structure. All of them solved the game by converting it to a reduced ordinary game. The general index case was studied in [26], where the theory of projector chains is used to decouple algebraic and differential parts of the descriptor system, and then the usual theory of ordinary differential games is applied to derive both necessary and sufficient conditions for the existence of feedback Nash equilibria for linear quadratic differential games. Moreover, the open-loop framework of soft-constrained descriptor differential game was studied in [20], also with its application in robust optimal control design.

This paper is the continuation of the work of [7] and [1], where the general linear quadratic differential game was considered for descriptor systems of index one. [1] have studied the soft-constrained ordinary differential game while [7] have studied the feedback (hard-constrained) descriptor differential game with an infinite planning horizon. By merging results from [1] and [7], in this paper, we study the feedback soft-constrained zero-sum descriptor differential game. The problem addressed in this paper is to find the smallest constraint value of $\gamma > 0$ -that appears in the cost function of the game (see e.g equation (2))- under which the game still has a Nash equilibrium and then to find the corresponding controllers for both players. We consider the game for both a finite and infinite planning horizon. We assume that players act non-cooperatively and the information they have is the current value of the state. We solve the problem by changing the descriptor differential game into a reduced ordinary differential game using the results in [6] and [7]. Like in [7] in this paper we try to provide complete parametrization of all feedback saddle point solution in term of descriptor systems. A different approach for such problem has been done by [27] where the problem is solved directly without modifying into ordinary game.

The feedback information structure implies that the resulting equilibrium actions have the important property that they are strongly time consist. That is, the equilibrium solution of the truncated game also remains an equilibrium solution for all consistent initial conditions x_{t_1} (that can be attained at t_1 from some consistent initial state at $t = 0$ for every $t_1 \in (0, t_f)$) off the equilibrium path. This property is, e.g., not satisfied by equilibrium actions constructed under an open-loop information structure [5].

[1]Index roughly translates to the number of differentiations required to represent a differential algebraic equation as a differential equation.

This paper is going to be organized as follows. Section 2 will include some basic results of linear quadratic differential games for descriptor systems and also state the main problem of this paper. Section 3 will present the main results for the soft-constrained zero-sum game both on a finite and infinite planning horizon as well as both for $\gamma = 1$ and $\gamma \neq 1$. Section 4 will illustrate in an example some results from the previous sections. At last, section 5 will conclude.

2. Preliminaries

In this paper we consider a game modeled by the set of coupled differential and algebraic equations

$$E\dot{x}(t) = Ax(t) + B_1u_1(t) + B_2u_2(t), \; x(0) = x_0 \tag{1}$$

where $E, A \in \mathbb{R}^{(n+r)\times(n+r)}$, $\operatorname{rank}(E) = n$, $B_i \in \mathbb{R}^{(n+r)\times m_i}$ and x_0 is the consistent initial state [2] (see (10) for a characterization). Vectors $u_i \in U_s$ are the actions player i can use to control the system, where U_s represents the set of all admissible actions for both players. The first player (minimizer) has a quadratic cost functional J_γ given by

$$J_\gamma(u_1,u_2) = \int_0^{t_f} \left[x^T(t)\bar{Q}x(t) + u_1^T(t)\bar{R}_1u_1(t) - \gamma u_2^T(t)\bar{R}_2u_2(t)\right] dt + x^T(t_f)\bar{Q}_{t_f}x(t_f) \tag{2}$$

where the parameter $\gamma \in \mathbb{R}$ is a weighting for the action of the second player who likes to maximize J_γ (or, stated differently, to minimize $-J_\gamma$). The game defined by (1,2) is called the (zero-sum) soft-constrained differential game. This terminology is used to capture the feature that in this game there is no hard bound with respect to u_2 [1].

We start this section by stating some required basic results. First, we recall from [2] some results concerning the descriptor system

$$E\dot{x}(t) = Ax(t) + f(t), \;\; x(0) = x_0 \tag{3}$$

and the associated matrix pencil

$$\lambda E - A. \tag{4}$$

System (3) and (4) are said to be regular if the characteristic polynomial is not identically zero. System (3) has a unique solution for any consistent initial state if and only if it is regular. Then, from [9] we recall the so-called Weierstrass' canonical form.

Theorem 2 *If* (4) *is regular, then there exist nonsingular matrices X and Y such that*

$$Y^TEX = \begin{bmatrix} I_n & 0 \\ 0 & N \end{bmatrix} \;\; and \;\; Y^TAX = \begin{bmatrix} A_1 & 0 \\ 0 & I_r \end{bmatrix} \tag{5}$$

[2] An initial state is called consistent if with this choice of the initial state the system (1) has a solution.

where A_1 is a matrix in Jordan form whose elements are the finite eigenvalues, $I_k \in \mathbb{R}^{k\times k}$ is the identity matrix and N is a nilpotent matrix also in Jordan form. A_1 and N are unique up to permutation of Jordan blocks.

Then, throughout this paper the next assumptions are made w.r.t system (1) (see also [7] and [14]):

1. matrix E is singular
2. $\det(\lambda E - A) \neq 0$
3. system (1) impulse controllable
4. matrix $N = 0$ in (5).

Assumption 4. implies that system (1) has index one.

Following the procedure in [21] and [22] yield that $(u_1^*(t), u_2^*(t))$ is a saddle point solution for the differential game (1,2) if and only if $(u_1^*(t), u_2^*(t))$ is a saddle point solution for the differential game defined by dynamical system

$$\dot{x}_1(t) = A_1 x_1(t) + Y_1 B_1 u_1(t) + Y_1 B_2 u_2(t), \;\; x_1(0) = \begin{bmatrix} I_n & 0 \end{bmatrix} X^{-1} x_0 \tag{6}$$

with the cost function for first player

$$J_\gamma(u_1, u_2) = \int_0^{t_f} \left\{ z^T(t) M_\gamma z(t) \right\} dt + x_1^T(t_f) Q_{t_f} x_1(t_f) \tag{7}$$

where $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = X^{-1}x(t)$, $x_1(t) \in \mathbb{R}^n$, $x_2(t) \in \mathbb{R}^r$, $z^T(t) = \begin{bmatrix} x_1^T(t) & u_1^T(t) & u_2^T(t) \end{bmatrix}$ and

$$M_\gamma = \begin{bmatrix} Q & V & W \\ V^T & R_{\bar{1}\bar{1}} & N \\ W^T & N^T & R_{\bar{2}\bar{2}\gamma} \end{bmatrix}. \tag{8}$$

The spellings of the matrices defined in (8) and other additional notation that will be used throughout this paper are presented in the Appendix.

In this feedback information framework we assume that the controls given by both players are in linear feedback control form defined by

$$u_i(t) = F_i(t) x(t) \in U_s, \;\; i = 1, 2, \tag{9}$$

where $F_i(t)$ is a piecewise continuous function. Furthermore, as has been discussed in [7], the set of consistent initial states for system (1) is

$$\left\{ x_1(0), x_2(0) \,\middle|\, x_2(0) = S^{-1} \begin{bmatrix} B_{12} & B_{22} \end{bmatrix} F X_1 x_1(0), \; x_1(0) \in \mathbb{R}^n \right\}. \tag{10}$$

Following the procedure in [7], we have that $(u_1^*(t), u_2^*(t))$ is a saddle point solution for the differential game (6,7) if and only if $\left(\widetilde{F}_1^*(t), \widetilde{F}_2^*(t)\right)$ is a saddle point solution for the differential game defined by the dynamical system

$$\dot{x}_1(t) = \left(A_1 + \begin{bmatrix} B_{11} & B_{21} \end{bmatrix} \begin{bmatrix} \widetilde{F}_1(t) \\ \widetilde{F}_2(t) \end{bmatrix}\right) x_1(t), \quad x_1(0) = \begin{bmatrix} I_n & 0 \end{bmatrix} X^{-1} x_0 \tag{11}$$

with cost function for the first player

$$J_\gamma\left(\widetilde{F}_1, \widetilde{F}_2\right) = \int_0^{t_f} \left\{ x_1^T(t) \begin{bmatrix} I & \widetilde{F}_1^T & \widetilde{F}_2^T \end{bmatrix} \widetilde{M}_\gamma \begin{bmatrix} I \\ \widetilde{F}_1 \\ \widetilde{F}_2 \end{bmatrix} x_1(t) \right\} dt + x_1^T(t_f) Q_{t_f} x_1(t_f) \tag{12}$$

where

$$\widetilde{M}_\gamma = \begin{bmatrix} \widetilde{Q} & \widetilde{V} & \widetilde{W} \\ \widetilde{V}^T & \widetilde{R}_{11} & \widetilde{N} \\ \widetilde{W}^T & \widetilde{N}^T & \widetilde{R}_{22\gamma} \end{bmatrix}.$$

Now, we define our main object of study in this paper, the feedback saddle point (FSP) equilibrium [6], [5].

Definition 1 $(F_1^*(t), F_2^*(t)) \in U_s$ *is a FSP equilibrium for the differential game* (1,2) *if for every* $(F_1(t), F_2^*(t)), (F_1(t), F_2^*(t)) \in U_s$,

$$J_\gamma(F_1^*(t), F_2(t)) \leqslant J_\gamma(F_1^*(t), F_2^*(t)) \leqslant J_\gamma(F_1(t), F_2^*(t)).$$

Then, the addressed problem in this paper is to find the set of γ such that the differential game (1,2) has a FSP solution $u_i(t) = F_i(t)x(t)$. Furthermore, in case the FSP solution exists, we want to characterize the set of FSP solutions. This, both for a finite and infinite planning horizon.

3. Soft-constrained linear quadratic descriptor differential game

3.1. Finite planing horizon

In this section we will characterize the set of FSP solutions for the game (1,2) using the reduced ordinary differential game described by the dynamical system (11) with the cost function (12). For $\gamma = 1$ the problem reduces to find the FSP solutions for the game

$$J_1\left(\widetilde{F}_1, \widetilde{F}_2\right) = \int_0^{t_f} \left\{ x_1^T(t) \begin{bmatrix} I & \widetilde{F}_1^T & \widetilde{F}_2^T \end{bmatrix} \widetilde{M} \begin{bmatrix} I \\ \widetilde{F}_1 \\ \widetilde{F}_2 \end{bmatrix} x_1(t) \right\} dt + x_1^T(t_f) Q_{t_f} x_1(t_f) \tag{13}$$

where

$$\widetilde{M} = \begin{bmatrix} \widetilde{Q} & \widetilde{V} & \widetilde{W} \\ \widetilde{V}^T & \widetilde{R}_{11} & \widetilde{N} \\ \widetilde{W}^T & \widetilde{N}^T & \widetilde{R}_{22} \end{bmatrix}.$$

From [7] one straightforwardly derives the next result.

Theorem 3 *Assume that $\gamma = 1$, matrix $\widetilde{G}$ is invertible, the matrices $\widetilde{Q}$ and Q_{t_f} are symmetric and $\widetilde{R}_{ii} > 0,\ i = 1,2$. Then $(F_1(t), F_2(t))$ is an FSP equilibrium for (1,2) for every consistent initial state if and only if $F_i(t) = \widetilde{F}_i(t)O^+(t) + Z_i(I - O(t)O^+(t))$, where $Z_i \in \mathbb{R}^{m_i\times(n+r)}$, $O(t) = X\begin{bmatrix} I \\ -B_{12}\widetilde{F}_1(t) - B_{22}\widetilde{F}_2(t)\end{bmatrix}$, O^+ denotes the Moore-Penrose inverse of O and $\left(\widetilde{F}_1(t), \widetilde{F}_2(t)\right)$ are given by*

$$\begin{bmatrix} \widetilde{F}_1(t) \\ \widetilde{F}_2(t) \end{bmatrix} = -\widetilde{G}^{-1}\begin{bmatrix} B_{11}^T K_1(t) + \widetilde{V}_1^T \\ -B_{21}^T K_1(t) - \widetilde{W}_1^T \end{bmatrix}$$

where $K_1(t)$ is the symmetric solution of the Riccati differential equation (RDE) on $[0, t_f]$

$$\begin{aligned} \dot{K}(t) = {} & -A_1^T K(t) - K(t)A_1 - \widetilde{Q} + \begin{bmatrix} \widetilde{V} + K(t)B_{11} & \widetilde{W} + K(t)B_{21} \end{bmatrix}\widetilde{G}^{-1} \\ & \times \begin{bmatrix} \widetilde{V} + K(t)B_{11} & -\left(\widetilde{W} + K(t)B_{21}\right) \end{bmatrix}^T, \qquad K(t_f) = \bar{Q}_{t_f}. \end{aligned} \tag{14}$$

Moreover, the saddle point value of the game is $L_1 = x_0^T X^{-T}\begin{bmatrix} I & 0 \end{bmatrix}^T K_1 \begin{bmatrix} I & 0 \end{bmatrix} X^{-1}x_0$.

Next, for $\gamma \neq 1$, in order to guarantee the differential game (1,2) has an FSP solution we must assure that the following Riccati differential equation

$$\begin{aligned} \dot{K}(t) = {} & -A_1^T K(t) - K(t)A_1 - \widetilde{Q} + \begin{bmatrix} \widetilde{V} + K(t)B_{11} & \widetilde{W} + K(t)B_{21} \end{bmatrix}\widetilde{G}_\gamma^{-1} \\ & \times \begin{bmatrix} \widetilde{V} + K(t)B_{11} & -\left(\widetilde{W} + K(t)B_{21}\right) \end{bmatrix}^T, \qquad K(t_f) = \bar{Q}_{t_f}. \end{aligned} \tag{15}$$

does not have a conjugate point on $[0, t_f]$. Following [1], the next lemma is useful to characterize when the Riccati differential equation (15) has a solution.

Lemma 1 *For γ large enough, the Riccati differential equation* (15) *has a solution on* $[0, t_f]$.

Therefore, it makes sense to introduce the following nonempty set

$$\Gamma_{CL} = \{\inf\Gamma_1,\ \inf\Gamma_2\}, \quad \hat{\gamma}_{CL} = \max\Gamma_{CL} \tag{16}$$

where

$$\Gamma_1 = \left\{ \widetilde{\gamma} > 0 \,\middle|\, \forall \gamma > \widetilde{\gamma},\ \widetilde{R}_{22\gamma} > 0 \right\}$$
$$\Gamma_2 = \left\{ \widetilde{\gamma} > 0 \,\middle|\, \forall \gamma > \widetilde{\gamma},\ \text{the RDE (15) does not have a conjugate point on } [0, t_f] \right\}.$$

For $\gamma = \hat{\gamma}^{CL}$ we have the next two lemmas [1].

Lemma 2 *For $\gamma = \hat{\gamma}^{CL}$ the Riccati differential equation* (15) *has a conjugate point.*

Lemma 3 *For any fixed $x_0 \in \mathbb{R}^{n+r}$, the function $\min\limits_{F_1} J_\gamma(F_1, F_2)$ has a finite supremum in $F_2 \in U_s$ if $\gamma > \hat{\gamma}^{CL}$, and only if $\gamma \geqslant \hat{\gamma}^{CL}$.*

Lemma 1 to 3 above along with Theorem 3 in [7] can be used to arrive at the following theorem (see also [1] for the ordinary game).

Theorem 4 *Consider the linear quadratic zero-sum soft-constrained differential game with a feedback information structure* (1,2) *defined on the interval $[0, t_f]$, assume that $\widetilde{R}_{11},\ \widetilde{R}_{22\gamma} > 0$ and let the parameter $\hat{\gamma}^{CL}$ be as defined by* (16)*. Then:*

1. *For $\gamma > \hat{\gamma}^{CL}$, the Riccati differential equation* (15) *does not have a conjugate point on the interval $[0, t_f]$.*
2. *For $\gamma > \hat{\gamma}^{CL}$, the differential game* (1,2) *admits an FSP solution, which is given by*

$$F_i(t) = \widetilde{F}_i(t) O^+(t) + Z_i \left(I - O(t) O^+(t) \right) \tag{17}$$

where $Z_i \in \mathbb{R}^{m_i \times (n+r)}$,
$O(t) = X \begin{bmatrix} I \\ -B_{12}\widetilde{F}_1(t) - B_{22}\widetilde{F}_2(t) \end{bmatrix}$ *and $\left(\widetilde{F}_1(t), \widetilde{F}_2(t)\right)$ are given by*

$$\begin{bmatrix} \widetilde{F}_1(t) \\ \widetilde{F}_2(t) \end{bmatrix} = -\widetilde{G}_\gamma^{-1} \begin{bmatrix} B_{11}^T K_{1_\gamma}(t) + \widetilde{V}_1^T \\ -B_{21}^T K_{1_\gamma}(t) - \widetilde{W}_1^T \end{bmatrix}, \tag{18}$$

where $K_{1_\gamma}(t)$ is the symmetric solution of the Riccati differential equation (15).

3. *For $\gamma > \hat{\gamma}^{CL}$, the saddle point value of the game is*

$$L_\gamma = x_0^T X^{-T} \begin{bmatrix} I & 0 \end{bmatrix}^T K_{1_\gamma}(0) \begin{bmatrix} I & 0 \end{bmatrix} X^{-1} x_0.$$

4. *If $\widetilde{R}_{22\hat{\gamma}^{CL}} > 0$, for $\gamma < \hat{\gamma}^{CL}$ the differential game has an unbounded upper value for all $F_1 \in U_s$ as well as an unbounded lower value.*

Proof Part 1 of the theorem follows from the definition of (16). Next consider part 2. Since (15) does not have a conjugate point on the interval $[0,t_f]$ then there exists a $K_{1_\gamma}(t)$ as a solution of (15). By Theorem 3 in [7] we have then that (18) is the unique FBSP solution for the reduced game (11,12). From the relationship

$$\widetilde{F}_i := F_i X \begin{bmatrix} I \\ H \end{bmatrix}, \tag{19}$$

we have then that the set of all FBSP solutions are given by (17). Part 3 follows directly from Theorem 3 in [7]. Finally, part 4 follows from part (ii) of Theorem 4.2 in [1]. □

3.2. Infinite planning horizon

In this section we consider the infinite planning horizon case, that is the game defined by dynamical system (1) with cost function, to be minimized for the first player,

$$J_\gamma^\infty(u,w) = \int_0^\infty \left[x^T(t)\bar{Q}x(t) + u^T(t)\bar{R}_1 u(t) - \gamma w^T(t)\bar{R}_2 w(t)\right] dt \tag{20}$$

or, equivalently,

$$J_\gamma^\infty\left(\widetilde{F}_1,\widetilde{F}_2\right) = \int_0^\infty \left\{ x_1^T(t) \begin{bmatrix} I & \widetilde{F}_1^T & \widetilde{F}_2^T \end{bmatrix} \widetilde{M}_\gamma \begin{bmatrix} I \\ \widetilde{F}_1 \\ \widetilde{F}_2 \end{bmatrix} x_1(t) \right\} dt \tag{21}$$

and for the second player $-J_\gamma^\infty$.

For well-posedness sake of the cost functional we restrict, in this section we also restrict the controller in the sense that it must stabilize the system for all consistent initial states. As discussed in [7] we assume that the feedback matrix F belongs to the set

$$\mathcal{F} := \left\{ F = \begin{bmatrix} F_1^T & F_2^T \end{bmatrix} \,\middle|\, \begin{array}{l} \text{all finite eigenvalues of } (E, A+BF) \text{ are stable} \\ \text{and } (E, A+BF) \text{ has index one,} \end{array} \right\} \tag{22}$$

where $B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$. That is, we assume that the matrix pairs (A_1, B_{i1}), $i = 1,2$, are stabilizable. So, in principle, each player is capable to stabilize the first part of the transformed system on his own. Furthermore, to ensure that such a stabilizing control for player one is generated naturally through the solution (optimization) process, unstable modes of A_1 will have to be observable through $\widetilde{Q}$. So, we make the assumption that the pair $\left(A_1, \widetilde{Q}^{\frac{1}{2}}\right)$ is detectable [1].

For $\gamma = 1$ we get the special form for cost function (21):

$$J^{\infty}\left(\widetilde{F}_1,\widetilde{F}_2\right)=\int_0^{\infty}\left\{x_1^T(t)\begin{bmatrix} I & \widetilde{F}_1^T & \widetilde{F}_2^T \end{bmatrix}\widetilde{M}\begin{bmatrix} I \\ \widetilde{F}_1 \\ \widetilde{F}_2 \end{bmatrix}x_1(t)\right\}dt. \tag{23}$$

Instead of the Riccati differential equation (14) we consider the following associated algebraic Riccati equation which will play an important role in this section

$$A_1^T K_1+K_1A_1+\widetilde{Q}_1-\left[\widetilde{V}_1+K_1B_{11}\widetilde{W}_1+K_1B_{21}\right]\widetilde{G}^{-1}\left[\widetilde{V}_1+K_1B_{11}-\left(\widetilde{W}_1+K_1B_{21}\right)\right]^T=0. \tag{24}$$

From [7] we obtain straightforwardly the following theorem.

Theorem 5 *Assume that $\gamma = 1$, matrix $\widetilde{G}$ is invertible and the matrices $\widetilde{R}_{ii} > 0$, $i = 1,2$. Then (F_1, F_2) is an FSP equilibrium for (1,2) for every consistent initial state if and only if $F_i = \widetilde{F}_i O^+ + Z_i(I - OO^+)$, where $Z_i \in \mathbb{R}^{m_i\times(n+r)}$, $O = X\begin{bmatrix} I \\ -B_{12}\widetilde{F}_1 - B_{22}\widetilde{F}_2 \end{bmatrix}$ and $\left(\widetilde{F}_1,\widetilde{F}_2\right)$ are given by*

$$\begin{bmatrix} \widetilde{F}_1 \\ \widetilde{F}_2 \end{bmatrix} = -\widetilde{G}^{-1}\begin{bmatrix} B_{11}^T K_1 + \widetilde{V}_1^T \\ -B_{21}^T K_1 - \widetilde{W}_1^T \end{bmatrix},$$

where K_1 is the symmetric solution of the algebraic Riccati equation (24) such that

$$A_1 - \begin{bmatrix} B_{11} & B_{21} \end{bmatrix}\widetilde{G}^{-1}\begin{bmatrix} B_{11}^T K_1 + \widetilde{V}_1^T \\ -B_{21}^T K_1 - \widetilde{W}_1^T \end{bmatrix}$$

is stable. Moreover, the cost for player one is $L_1 = x_0^T X^{-T}\begin{bmatrix} I & 0 \end{bmatrix}^T K_1 \begin{bmatrix} I & 0 \end{bmatrix} X^{-1}x_0$ and for player two is $-L_1$.

To prove for $\gamma \neq 1$ in the next theorems, let us introduce the counterpart of (16)

$$\Gamma_{CL}^{\infty} = \left\{\inf\Gamma_1^{\infty},\ \inf\Gamma_2^{\infty}\right\},\ \ \gamma^{\infty} = \max\Gamma_{CL}^{\infty}, \tag{25}$$

where

$$\Gamma_1^{\infty} = \left\{\widetilde{\gamma} > 0 \,\middle|\, \forall\gamma \geqslant \widetilde{\gamma},\ \widetilde{R}_{22\gamma} > 0\right\},$$
$$\Gamma_2^{\infty} = \left\{\widetilde{\gamma} > 0 \,\middle|\, \forall\gamma \geqslant \widetilde{\gamma},\ \bar{J}_{\gamma}^{\infty} := \inf_{F_1\in U_s}\sup_{F_2\in U_s} J_{\gamma}^{\infty}(F_1,F_2) \leqslant 0\right\}.$$

Here $\bar{J}_{\gamma}^{\infty}$ denotes the upper value of the game (1,20). We also introduce the following algebraic Riccati equation corresponding to the Riccati differential equation (15)

$$A_1^T K_1+K_1A_1+\widetilde{Q}_1-\left[\widetilde{V}_1+K_1B_{11}\widetilde{W}_1+K_1B_{21}\right]\widetilde{G}_{\gamma}^{-1}\left[\widetilde{V}_1+K_1B_{11}-\left(\widetilde{W}_1+K_1B_{21}\right)\right]^T=0. \tag{26}$$

The next theorem characterizes when the game defined by (11,21) has a finite upper value if $Q \geqslant 0$ (see [1] for the proof).

Theorem 6 *Let* (A_1, B_{i1}), $i = 1,2$ *be stabilizable and* $\left(A_1, \widetilde{Q}^{\frac{1}{2}}\right)$ *be detectable. Then, there exists a finite scalar* $\gamma^{\infty} > 0$ *such that for all* $\gamma > \gamma^{\infty}$ *the game defined by* (11,21) *has finite upper value, i.e.,*

$$\bar{J}_{\gamma}^{\infty}(x_0) < \infty, \text{ for all } x_0 \in \mathbb{R}^{n+r}, \tag{27}$$

and there exists a nonnegative definite solution to (26)*, say* $K_{1\gamma}^{+}$*, with the further property that*

$$\hat{F}_{\gamma}^{+} := A_1 - \begin{bmatrix} B_{11} & B_{21} \end{bmatrix} \widetilde{G}_{\gamma}^{-1} \begin{bmatrix} B_{11}^T K_{1\gamma}^{+} + \widetilde{V}_1^T \\ -B_{21}^T K_{1\gamma}^{+} - \widetilde{W}_1^T \end{bmatrix} \tag{28}$$

is stable.
Such solution $K_{1\gamma}^{+}$ *has the following additional properties (for all* $\gamma > \gamma^{\infty}$ *):*

1. $K_{1\gamma}^{+} > 0$*, if* $\left(A_1, \widetilde{Q}^{\frac{1}{2}}\right)$ *is observable;*
2. $K_{1\gamma}^{+}$ *is the unique solution of* (26) *in the class of nonnegative definite matrices that satisfy* (28)*;*
3. *In the class of nonnegative definite matrices,* $K_{1\gamma}^{+}$ *is the minimal solution of* (26)*;*
4. *The matrix*
$$\hat{A}_{1\gamma} := A_1 - \begin{bmatrix} B_{11} & 0 \end{bmatrix} \widetilde{G}_{\gamma}^{-1} \begin{bmatrix} B_{11}^T K_{1\gamma}^{+} + \widetilde{V}_1^T \\ 0 \end{bmatrix} \tag{29}$$
is stable;
5. $\bar{J}_{\gamma}^{\infty}(x_0) := \sup_{\widetilde{F}_2 \in U_s} J_{\gamma}^{\infty}\left(\widetilde{F}_1^{*}, \widetilde{F}_2\right) = x_0^T X^{-T} \begin{bmatrix} I & 0 \end{bmatrix}^T K_{1\gamma}^{+} \begin{bmatrix} I & 0 \end{bmatrix} X^{-1} x_0$*, where*
$$\widetilde{F}_1^{*} = -\begin{bmatrix} I & 0 \end{bmatrix} \widetilde{G}_{\gamma}^{-1} \begin{bmatrix} B_{11}^T K_{1\gamma}^{+} + \widetilde{V}_1^T \\ -B_{21}^T K_{1\gamma}^{+} - \widetilde{W}_1^T \end{bmatrix}; \tag{30}$$
6. $K_{1\gamma}^{+} = \lim_{t_f \to \infty} K_{1\gamma}(t; t_f)$ *for* $Q_{t_f} = 0$.

If $\gamma < \gamma^{\infty}$*, the upper value is infinite, and* (26) *has no real solution which also satisfies property 4.*

Next, the assumption of detectability of $\left(A_1, \widetilde{Q}^{\frac{1}{2}}\right)$ can be relaxed by requiring that the system trajectory be asymptotically stable [1]. We present this consequence in the following theorem.

Theorem 7 *Let (A_1, B_{i1}), $i = 1,2$ be stabilizable, $Q \geqslant 0$ and $\left(A_1, \widetilde{Q}^{\frac{1}{2}}\right)$ have no unobservable modes on the imaginary axis. Then, there exists a finite scalar $\gamma^\infty > 0$, such that for all $\gamma > \gamma^\infty$ the game in which player one is restricted to stabilizing controllers has a finite upper value, i.e.,*

$$\bar{J}_\gamma^\infty (x_0) := \inf_{\widetilde{F}_1 \in U_s : x(\infty)=0} \sup_{\widetilde{F}_2 \in U_s} J_\gamma^\infty \left(\widetilde{F}_1, \widetilde{F}_2\right) < \infty, \text{ for all } x_0 \in \mathbb{R}^{n+r},$$

and there exists a nonnegative definite solution to (26)*, say $\widetilde{K}_{1\gamma}$, with the further property that*

$$\hat{\widetilde{F}}_\gamma := A_1 - \begin{bmatrix} B_{11} & B_{21} \end{bmatrix} \widetilde{G}_\gamma^{-1} \begin{bmatrix} B_{11}^T \widetilde{K}_{1\gamma} + \widetilde{V}_1^T \\ -B_{21}^T \widetilde{K}_{1\gamma} - \widetilde{W}_1^T \end{bmatrix} \tag{31}$$

is stable.
Such solution $\widetilde{K}_{1\gamma}$ has the following additional properties (for all $\gamma > \gamma^\infty$):

1. $\widetilde{K}_{1\gamma} = \lim_{t_f \to \infty} K_{1\gamma}(t;t_f)$, for $Q_{t_f} = \widetilde{\Sigma}$*, where $\widetilde{\Sigma}$ is the unique maximal solution of the algebraic Riccati equation associated with the limiting control problem*

$$A_1^T \Sigma + \Sigma A_1 - (\Sigma Y_1 B_1 + V_1) R_{11}^{-1} \left(B_1^T Y_1^T \Sigma + V_1^T\right) + Q = 0; \tag{32}$$

2. $\widetilde{K}_{1\gamma}$ *is the unique solution of* (26) *in the class of nonnegative definite matrices that satisfy* (31)*;*

3. *In the class of nonnegative definite matrices K_1 with the property $K_1 \geqslant \widetilde{\Sigma}$, $\widetilde{K}_{1\gamma}$ is the minimal solution of (26);*

4. *The matrix $\widetilde{A}_{1\gamma} := A_1 - B_{11} \widetilde{G}_\gamma^{-1} \left[B_{11}^T \widetilde{K}_{1\gamma} + \widetilde{V}_1^T\right]$ is stable;*

5. $\bar{J}_\gamma^\infty := \sup_{\widetilde{F}_2 \in U_s} J_\gamma^\infty \left(\widetilde{F}_1^*, \widetilde{F}_2\right) = x_0^T X^{-T} \begin{bmatrix} I & 0 \end{bmatrix}^T \widetilde{K}_{1\gamma} \begin{bmatrix} I & 0 \end{bmatrix} X^{-1} x_0$*, where*

$$\widetilde{F}_1^* = -\begin{bmatrix} I & 0 \end{bmatrix} \widetilde{G}_\gamma^{-1} \begin{bmatrix} B_{11}^T \widetilde{K}_{1\gamma} + \widetilde{V}_1^T \\ -B_{21}^T \widetilde{K}_{1\gamma} - \widetilde{W}_1^T \end{bmatrix}.$$

If $\gamma < \gamma^\infty$, the upper value is infinite, and (26) *has no real solution which also satisfies property 4.*

Next, we characterize FSP solutions for the infinite-horizon linear quadratic soft-constrained differential game (1,20). One of the interesting questions in this situation is if (15) does not have a conjugate point on $[0,t_f]$ for any t_f and in case $\lim_{t_f\to\infty} K_1(t;t_f) = \bar{K}_1$ exists, does this necessarily imply that the pair

$$\begin{bmatrix} \widetilde{F}_1 \\ \widetilde{F}_2 \end{bmatrix} = -\widetilde{G}_\gamma^{-1} \begin{bmatrix} B_{11}^T \bar{K}_1 + \widetilde{V}_1^T \\ -B_{21}^T \bar{K}_1 - \widetilde{W}_1^T \end{bmatrix} \tag{33}$$

is in saddle point equilibrium? [1] has shown that there is no continuity in the saddle point property of the maximizers feedback policy, as $t_f \to \infty$, whereas there is continuity in the value of the game and in the feedback policy of the minimizer. Applying this fact and Theorem 5 to the infinite-horizon linear quadratic soft-constrained differential game (1,20) yields then the following theorem (see [1] for the ordinary game).

Theorem 8 *Consider the infinite-horizon linear quadratic soft-constrained differential game* (1,20) *with a feedback information structure. Assume an arbitrary consistent initial state* $x_0 \in \mathbb{R}^{n+r}$, $Q_{t_f} = 0$. *Moreover assume that* $\widetilde{R}_{11}$, $\widetilde{R}_{22\gamma} > 0$ *and the pair* $\left(A_1, \widetilde{Q}^{\frac{1}{2}}\right)$ *is detectable. Then:*

1. *For each fixed t, the solution to* (15), $K_1(t;t_f)$, *is nondecreasing in* t_f, *that is if* (15) *has no conjugate point in a given interval* $[0,t_f]$, *then*

$$K_1\left(t;t_f^{`}\right) - K_1\left(t;t_f^{``}\right) \geqslant 0,\ \ t_f > t_f^{`} > t_f^{``} \geqslant 0;$$

2. *If there exists a nonnegative definite solution of* (26)*, there is a minimal solution, denoted* $K_{1\gamma}^+$. *This matrix has the property that* $K_{1\gamma}^+ - K_1\left(t;t_f\right) \geqslant 0$ *for all* $t_f \geqslant 0$, *where* $K_1\left(t;t_f\right)$ *is the solution of* (15) *with* $Q_{t_f} = 0$. *If* $\left(A_1, \widetilde{Q}^{\frac{1}{2}}\right)$ *is observable, then every nonnegative definite solution of* (26) *is positive definite;*

3. *The differential game* (1,20) *has equal upper and lower values if and only if the algebraic Riccati equation* (26) *admits a nonnegative definite solution, in which case the common value is*

$$L_\gamma^{\infty *} = x_0^T X^{-T} \begin{bmatrix} I & 0 \end{bmatrix}^T K_{1\gamma}^+ \begin{bmatrix} I & 0 \end{bmatrix} X^{-1} x_0;$$

4. *If the upper value is finite for some* $\gamma > 0$, *(say,* $\gamma = \gamma^\infty$*), then it is bounded and equal the lower value for all* $\gamma > \gamma^\infty$*;*

5. *If* $K_{1\gamma}^+ \geqslant 0$ *exists let* F_i^* *be given by*

$$F_i^* = \widetilde{F}_i^* O^+ + Z_i\left(I - OO^+\right), \tag{34}$$

where $Z_i \in \mathbb{R}^{m_i \times (n+r)}$, $O = X \begin{bmatrix} I \\ -B_{12}\widetilde{F}_1^* - B_{22}\widetilde{F}_2^* \end{bmatrix}$, $\widetilde{F}_1^*$ *and* $\widetilde{F}_2^*$ *are given by*

$$\begin{bmatrix} \widetilde{F}_1^* \\ \widetilde{F}_2^* \end{bmatrix} = -\widetilde{G}_\gamma^{-1} \begin{bmatrix} B_{11}^T K_{1\gamma}^+ + \widetilde{V}_1^T \\ -B_{21}^T K_{1\gamma}^+ - \widetilde{W}_1^T \end{bmatrix}. \tag{35}$$

Then F_1^* *is the steady-state feedback controller attains the finite upper value, in the sense that*

$$\sup_{F_2 \in U_s} J_\gamma^\infty (F_1^*, F_2) = L_\gamma^{\infty *} = x_0^T X^{-T} \begin{bmatrix} I & 0 \end{bmatrix}^T K_{1\gamma}^+ \begin{bmatrix} I & 0 \end{bmatrix} X^{-1} x_0, \tag{36}$$

and F_2^* *is the maximizing feedback solution in* (20).

6. *If the upper value is bounded for some* $\gamma > 0$, *(say,* $\gamma = \gamma^\infty$*), then for all* $\gamma > \gamma^\infty$, *the two feedback matrices* (28) *and* (29) *are asymptotically stable;*

7. *For* $\gamma > \gamma^\infty$, $K_{1\gamma}^+ \geqslant 0$ *is the unique solution of* (26) *in the class of nonnegative definite matrices which make* $\hat{F}_\gamma$ *stable;*

8. *Since* (A_1, B_{i1}), $i = 1,2$ *are stabilizable then the upper value is bounded for some finite* $\gamma > 0$.

Proof Similar as in the proof of Theorem 4, we can restrict for the proof of most statements to the reduced order system. For part 1 – 4 see part (i) – (iv) of Theorem 4.8 from [1]. Part 5 follows by using the relationship (19) again, using the results of Theorem 3 from [7]. Finally, part 6 – 8 result again, from part (vi) – (viii) of Theorem 4.8 in [1]. □

The more general version of Theorem 8 can be obtained by relaxing the detectability condition of $\left(A_1, \widetilde{Q}^{\frac{1}{2}}\right)$. For this condition by letting $\gamma^{-2} = 0$ in (26) we arrive at the standard algebraic Riccati equation (32) that arises in linear regulator theory for descriptor system (see also [1] for the ordinary game). Let $\widetilde{\Sigma}^+ \geqslant 0$ denote its maximal solution. We are now in position to state the following theorem (For a proof we refer to Theorem 4.8' in [1]).

Theorem 9 *Consider the framework of Theorem 8, but with* (A_1, B_{i1}), $i = 1,2$, *stabilizable and* $\left(A_1, \widetilde{Q}^{\frac{1}{2}}\right)$ *having no unobservable modes on the imaginary axis. Let* $\widetilde{\Sigma}^+$ *be the unique maximal solution of* (32). *Then:*

1. *There exists a finite* $\gamma > 0$, *(say,* $\gamma = \gamma^\infty$*), such that for all* $\gamma > \gamma^\infty$, *the upper value is bounded, and for* $\gamma < \gamma^\infty$, *it is unbounded;*

2. *For all $\gamma > \gamma^{\infty}$, in the class of all symmetric matrices bounded from below by $\widetilde{\Sigma}^{+}$, there is a minimal one that solves* (26)*, to be denoted $\widetilde{K}_{1\gamma}^{+}$. This matrix has the additional property that $\widetilde{K}_{1\gamma}^{+} - K_1\left(t;t_f\right) \geqslant 0$ for all $t_f \geqslant 0$, where $K_1\left(t;t_f\right)$ is the solution of* (14) *with $Q_{t_f} = \widetilde{\Sigma}^{+}$ and γ fixed;*

3. *For all $\gamma > \gamma^{\infty}$, the differential game has equal upper and lower value. The common value of the game is*

$$L_{\gamma}^{\infty *} = x_0^T X^{-T} \begin{bmatrix} I & 0 \end{bmatrix}^T \widetilde{K}_{1\gamma}^{+} \begin{bmatrix} I & 0 \end{bmatrix} X^{-1} x_0;$$

4. *For all $\gamma > \gamma^{\infty}$, the steady-state FSP equilibrium F_i^* given by $F_i^* = \widetilde{F}_i^* O^+ + Z_i\left(I - OO^+\right)$, where $Z_i \in \mathbb{R}^{m_i \times (n+r)}$, $O = X \begin{bmatrix} I \\ -B_{12}\widetilde{F}_1^* - B_{22}\widetilde{F}_2^* \end{bmatrix}$, F_1^* and $\widetilde{F}_2^*$ are given by* (35)*. F_1^* attains the finite upper value $L_{\gamma}^{\infty *}$, and*

$$\sup_{F_2 \in U_s} J_{\gamma}^{\infty}\left(F_1^*, F_2\right) = L_{\gamma}^{\infty *} = x_0^T X^{-T} \begin{bmatrix} I & 0 \end{bmatrix}^T \widetilde{K}_{1\gamma}^{+} \begin{bmatrix} I & 0 \end{bmatrix} X^{-1} x_0.$$

The maximizing feedback solution above then is F_2^*;

5. *For all $\gamma > \gamma^{\infty}$, the two matrices* (28) *and* (29) *are stable (again with $\widetilde{K}_{1\gamma}^{+}$ taken above);*

6. *For all $\gamma > \gamma^{\infty}$, the matrix $\widetilde{K}_{1\gamma}^{+}$ defined in 2. is the unique solution of* (26) *in the class of nonnegative definite matrices which make* (28) *stable;*

7. *For $\gamma < \gamma^{\infty}$, the algebraic Riccati equation* (26) *has no real solution that also makes* (29) *stable.*

4. Numerical example

Consider the zero-sum game between player 1 and player 2 defined by the system

$$E\dot{x}(t) = Ax(t) + B_1 u_1(t) + B_2 u_2(t), \quad x(0) = x_0 \tag{37}$$

and cost function to be minimized by player 1

$$J_{1\gamma}(u_1, u_2) = \int_0^{\infty} \left\{ x^T(t)\bar{Q}x(t) + u_1^T(t)\bar{R}_1 u_1(t) - \gamma u_2^T(t)\bar{R}_2 u_2(t) \right\} dt \tag{38}$$

where $E = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$, $B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\bar{Q} = \begin{bmatrix} 5 & 1 \\ 0 & 1 \end{bmatrix}$, $\bar{R}_1 = [1]$, $\bar{R}_2 = [3]$, $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \in \mathbb{R}^2$. Player 1 is the minimizing player which controls the slow dynamics of the systems (described by the state $x_1(t)$), using the control $u_1(t)$, whereas player 2 is the maximizing player who controls the fast dynamics of the systems (described by the state $x_2(t)$), using control $u_2(t)$. With $Y^T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ the matrix pencil (E, A) can be rewritten into its Weierstrass' canonical form (5) where $X_1 = Y_2^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $X_2 = Y_1^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $N = [0]$, and $J = [1]$. Then, the game can be described by (11,21) with matrix $\widetilde{M}_\gamma$ and $\widetilde{G}_\gamma$:

$$\widetilde{M}_\gamma = \begin{bmatrix} 1 & -3 & -3 \\ -3 & 14 & 13 \\ -3 & 13 & 11+\gamma \end{bmatrix} \text{ and } \widetilde{G}_\gamma = \begin{bmatrix} 14 & 13 \\ -13 & 11+\gamma \end{bmatrix},$$

respectively. Using (35) we have

$$\begin{bmatrix} \widetilde{F}_1^* \\ \widetilde{F}_2^* \end{bmatrix} = \begin{bmatrix} \frac{(24+\gamma)K_{1\gamma}^+ - 3\gamma - 72}{14\gamma - 15} \\ \frac{-K_{1\gamma}^+ + 3}{14\gamma - 15} \end{bmatrix}$$

where $K_{1\gamma}^+ = \frac{34\gamma + 114 - \sqrt{1176\gamma^2 + 7324\gamma - 7428}}{2\gamma + 46}$ is the nonnegative definite solution of the algebraic Riccati equation

$$-(\gamma + 23) K_{1\gamma}^{+2} + (34\gamma + 114) K_{1\gamma}^+ + (5\gamma - 222) = 0,$$

such that

$$\frac{(23+\gamma) K_{1\gamma}^+ + 11\gamma - 84}{14\gamma - 15} < 0. \tag{39}$$

Solving (39) above, we get $\gamma^\infty = \frac{15}{14} = 1.0714$. Furthermore, after some calculations, matrix O and O^+ result as

$$O = \begin{bmatrix} \frac{-(23+\gamma)K_{1\gamma}^+ + 17\gamma + 70}{14\gamma - 15} \\ -\frac{(23+\gamma)K_{1\gamma}^+ - 3\gamma - 85}{14\gamma - 15} \end{bmatrix} \text{ and } O^+ = \begin{bmatrix} \Theta_1 & \Theta_2 \end{bmatrix}$$

respectively, where

$$\Theta_1 = \frac{196\gamma^2 - 420\gamma + 225}{(\gamma^2 + \gamma + 529) K_{1\gamma}^{+2} - (6\gamma^2 + 69\gamma + 3979) K_{1\gamma}^+ + 205\gamma^2 + 90\gamma + 7550}$$

and

$$\Theta_2 = \frac{\left(-196\gamma^3+4928\gamma^2+9435\gamma-5175\right)K_{1\gamma}^{+}+2156\gamma^3-2422\gamma^2+4200\gamma+225}{(143\gamma^3-\gamma^2+7391\gamma-7935)K_{1\gamma}^{+2}-(84\gamma^3+856\gamma^2+54671\gamma-59685)K_{1\gamma}^{+}+2870\gamma^3-1815\gamma^2+93350\gamma+116250}.$$

If $\gamma = 3.5$, this yields the next set of equilibrium strategies (34) (see Theorem 8):

$$\begin{aligned}\begin{bmatrix} F_1^* \\ F_2^* \end{bmatrix} &= \begin{bmatrix} -0.4520 & -0.3216 \\ 0.0164 & 0.0117 \end{bmatrix} + \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}\begin{bmatrix} 0.3099 & -0.4910 \\ -0.4356 & 0.6901 \end{bmatrix} \\ &=: \begin{bmatrix} F_{11}^* & F_{12}^* \\ F_{21}^* & F_{22}^* \end{bmatrix}\end{aligned} \tag{40}$$

where $z_{ij} \in \mathrm{R}$, $\mathrm{i,j} = 1,2$.

Next, to analyze the robustness property of the strategies above, we consider the system if one uses the equilibrium actions F_i^* from (40) to control dynamical system (37). Then the closed-loop system is described by

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\dot{x}(t) = \begin{bmatrix} 1+F_{11}^*+2F_{21}^* & F_{11}^*+F_{12}^*+2F_{21}^*+2F_{22}^* \\ F_{11}^*+F_{21}^* & 1+F_{11}^*+F_{12}^*+F_{21}^*+F_{22}^* \end{bmatrix} x(t) =: A_{cl}x(t). \tag{41}$$

To increase robustness, the players have to choose the $z_{ij} \in \mathbb{R}$, $i,j = 1,2$ in such a way that the closed-loop system (41) becomes as stable as possible. Therefore, we need to determine $z_{ij} \in \mathbb{R}$, $i,j = 1,2$ in such a way that the real part of the largest root of the polynomial in λ

$$z(z_{ij}, i,j=1,2) := \left\{ \begin{vmatrix} \lambda-1-F_{11}^*-2F_{21}^* & -F_{11}^*-F_{12}^*-2F_{21}^*-2F_{22}^* \\ -F_{11}^*-F_{21}^* & -1-F_{11}^*-F_{12}^*-F_{21}^*-F_{22}^* \end{vmatrix} = 0 \right\}$$

is minimal. Elementary calculations show that

$$\lambda = \frac{(1+F_{11}^*+F_{12}^*+F_{21}^*+F_{22}^*)+(F_{11}^*+2F_{21}^*+F_{12}^*F_{21}^*-F_{11}^*F_{22}^*)}{1+F_{11}^*+F_{12}^*+F_{21}^*+F_{22}^*}.$$

So, to enforce a minimal real part of λ the next two conditions must be satisfied $1+F_{11}^*+F_{12}^*+F_{21}^*+F_{22}^* = -\varepsilon$ (where ε is a small positive number) and $F_{11}^*+2F_{21}^*+F_{12}^*F_{21}^*-F_{11}^*F_{22}^* > 0$. Then, as there is a lot of freedom in the choice of the z_{ij} values, by choosing z_{ij} accordingly the system becomes as stable as possible. Numerical simulations show that, choosing z_{11} and z_{21} as small as possible and z_{12} and z_{22} as big as possible, will attain a robust strategy for the game (37,38).

Another option to increase robustness occurs if the parameter $\gamma \in \Gamma_{CL}^{\infty}$ could be chosen freely by the players. Then, choosing $\gamma = \gamma^{\infty} + \varepsilon$ for ε sufficiently small will make the closed loop system as stable as possible. Figure 1 illustrates the optimal trajectory of

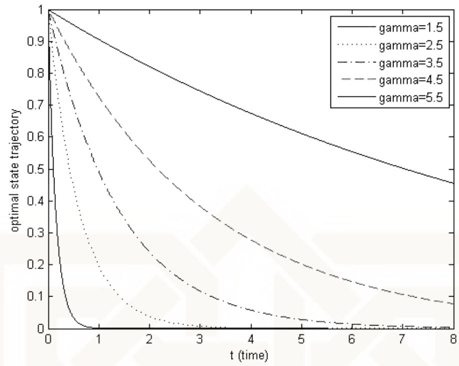

Figure 1. Optimal trajectory of state $x_1^*(t)$

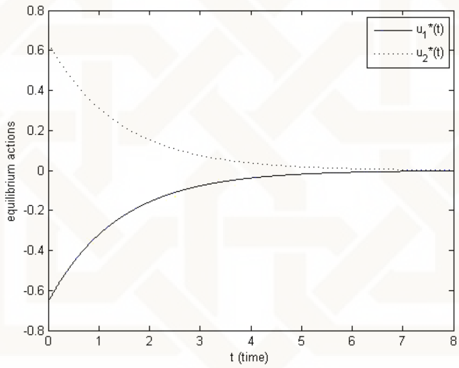

Figure 2. Equilibrium actions $\left(u_1^* = F_1^* x, u_2^* = F_2^* x\right)$

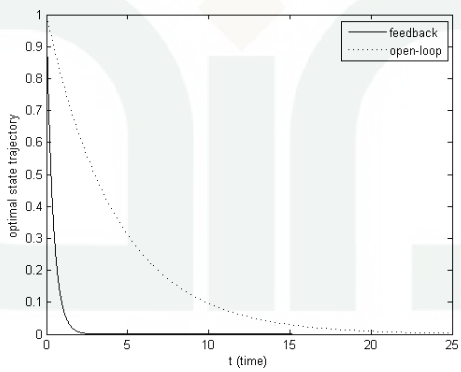

Figure 3. The comparison of the optimal trajectory uses open-loop and feedback controller

state $x_1^*(t)$ when player 1 uses the equilibrium strategies (40) for different values of γ. Figure 2 illustrates the equilibrium actions of the game used by both players for $\gamma = 3.5$. Compared to the open-loop controller, the feedback controller has advantages in terms of speed to reach stability. Based on the results obtained in [20], Figure 3 shows a comparison of the optimal trajectory obtained if the system uses the open-loop and feedback controller, respectively.

5. Concluding remarks

This paper studies the linear quadratic zero-sum soft-constrained differential game for descriptor systems which have index one. Necessary and sufficient conditions for the existence of an FSP equilibrium have been derived. The paper shows how the solution of the game depends on a Riccati differential equation for the finite horizon case and an algebraic Riccati equation for the infinite horizon. The paper also shows how the critical value $\hat{\gamma}^{CL}$ for finite planning horizon and γ^{∞} for infinite planning horizon play a key role. A numerical example illustrating some of the theoretical results is presented.

The problem addressed in this paper is restricted to index one descriptor systems. To find an FSP equilibrium in a zero-sum game that has higher order index is still an open problem to be analyzed.

Appendix

We use the next shorthand notation in this paper:

$$Q := X_1^T \bar{Q} X_1 =: \widetilde{Q}, \qquad V := -X_1^T \bar{Q} X_2 Y_2 B_1, \quad W := -X_1^T \bar{Q} X_2 Y_2 B_2,$$

$$N := B_1^T Y_2^T X_2^T \bar{Q} X_2 Y_2 B_2, \, \widetilde{V} := -X_1^T \bar{Q} X_2 B_{12} + X_1^T V, \, \widetilde{W} := -X_1^T \bar{Q} X_2 B_{22} + X_1^T W,$$

$$\widetilde{N} := B_{12}^T X_2^T \bar{Q} X_2 B_{22} - V^T X_2 B_{22} - B_{12}^T X_2^T W + N, \, R_{\bar{1}\bar{1}} := B_1^T Y_2^T X_2^T \bar{Q} X_2 Y_2 B_1 + \bar{R}_1,$$

$$\widetilde{R}_{11} := B_{12}^T X_2^T \bar{Q} X_2 B_{12} - V^T X_2 B_{12} - B_{12}^T X_2^T V + R_{11}, \, R_{\bar{2}\bar{2}\gamma} := B_2^T Y_2^T X_2^T \bar{Q} X_2 Y_2 B_2 - \gamma \bar{R}_2,$$

$$\widetilde{R}_{22} := B_{22}^T X_2^T \bar{Q} X_2 B_{22} - W^T X_2 B_{22} - B_{22}^T X_2^T W + R_{22},$$

$$\widetilde{R}_{22\gamma} := B_{22}^T X_2^T \bar{Q} X_2 B_{22} - W^T X_2 B_{22} - B_{22}^T X_2^T W + \gamma R_{22},$$

$$S := I + \begin{bmatrix} B_{12} & B_{22} \end{bmatrix} F X_2, \, F = \begin{bmatrix} F_1^T & F_2^T \end{bmatrix}, \, B_{i1} = \begin{bmatrix} I & 0 \end{bmatrix} Y^T B_i,$$

$$B_{i2} = \begin{bmatrix} 0 & I \end{bmatrix} Y^T B_i, \widetilde{F}_i := F_i X \begin{bmatrix} I \\ H \end{bmatrix}, H := -\left(I + \begin{bmatrix} B_{12} & B_{22} \end{bmatrix} F X_2\right)^{-1} \begin{bmatrix} B_{12} & B_{22} \end{bmatrix} F X_1,$$

$$\widetilde{G} = \begin{bmatrix} \widetilde{R}_{11} & \widetilde{N}_1 \\ -\widetilde{N}_1^T & \widetilde{R}_{22} \end{bmatrix}, \widetilde{G}_\gamma = \begin{bmatrix} \widetilde{R}_{11} & \widetilde{N}_1 \\ -\widetilde{N}_1^T & \widetilde{R}_{22\gamma} \end{bmatrix}.$$

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