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ON SEMISIMPLE LEAVITT PATH ALGEBRAS OVER A COMMUTATIVE UNITAL RING

by

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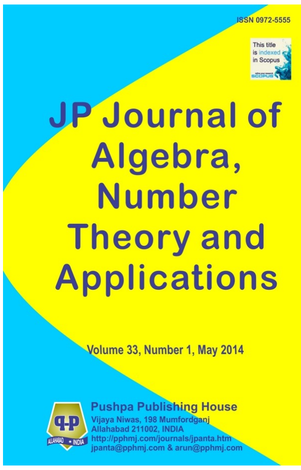

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ON SEMISIMPLE LEAVITT PATH ALGEBRAS OVER A COMMUTATIVE UNITAL RING

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Abstract

A finite acyclic graph always contains a sink, a vertex that does not emit edges. Any sink at the graph will generate minimal basic ideal of the Leavitt path algebra over a commutative unital ring. Moreover, the Leavitt path algebra on the finite acyclic graph is a direct sum of minimal basic ideals generated by the sinks. In other words, Leavitt path algebra over the commutative unital ring on the finite acyclic graph is basically semisimple, but not necessarily semisimple. The Leavitt path algebra is semisimple if and only if the commutative unital ring is semisimple.

1. Introduction

Algebraically, a (directed) graph $E = (E^0, E^1, s, r)$ is a pair of 4-tuple consisting of a set of vertices, a set of edges, and two mappings $s, r : E^1 \to E^0$, in which for every edge $e \in E^1$, $s(e), r(e) \in E^0$ are, respectively, the source and the end of e [4]. A path of length n in the graph

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E is a sequence of n edges. Let $\mu_n = e_1 e_2 \cdots e_n$ be such that $r(e_i) = s(e_{i+1})$ for $i = 1, 2, \ldots, n-1.$ A path is called a *cycle* if the end and the source of path are the same, but neither edge has the same source. A graph is *acyclic* if it has no cycle.

Leavitt has defined an extended graph of E as $\hat{E} = (E^0, E^1 \cup (E^1)^*, s', r'),$ where $(E^1)^*$ is a set of edges with direction opposite to the edges in $E^1,$ i.e., $(E^1)^* = \{e^* : e \in E^1\}.$ The edges in E^1 are named *real edges* and the edges in $(E^1)^*$ are called *ghost edges*. Leavitt path algebras over a field are extended path algebras over the field of the extended graph $\hat{E}$ satisfying Cuntz-Krieger conditions CK1 and CK2 [1, 3].

Tomforde [7] has generalized Leavitt path algebra over a commutative unital ring R, denoted by $L_R(E),$ as a free R-algebra that satisfies: (1) $s(e)e = er(e) = e, \forall e \in E^1,$ (2) $r(e)e^* = e^* r(e) = e^*, \forall e \in E^1,$ (3) $CK1:$ $e^* f = \delta_{e,f} r(e), \forall e, f \in E^1,$ and (4) $CK2: v = \sum_{\{e \in E^1, s(e) = v\}} ee^*, \forall v \in E^0, v$ is not a sink. Multiplication of nonzero monomials $\alpha\beta^*$ and $\gamma\delta^*$ in $L_R(E)$ is as follows:

$$(\alpha\beta^*)(\gamma\delta^*) = \begin{cases} \alpha\gamma'\delta^*, & \text{if } \gamma = \beta\gamma', \\ \alpha\delta^*, & \text{if } \gamma = \beta, \\ \alpha\beta'^*\delta^*, & \text{if } \beta = \gamma\beta', \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

In addition, an ideal I in $L_R(E)$ is a *basic ideal* if it satisfies: for every nonzero $c \in R,$ $u \in E^0$ if $cu \in I,$ then $u \in I$ [7]. Leavitt path algebra $L_R(E)$ is *basically simple* if the only basic ideals of $L_R(E)$ are $\{0\}$ and itself.

An algebra is semisimple if it is a direct sum of minimal ideals, i.e., an ideal that does not contain any nonzero ideal except itself [9]. Analogously, a minimal basic ideal in $L_R(E)$ is defined as a basic ideal that does not contain any nonzero basic ideal except itself. In addition, $L_R(E)$ is basically semisimple if it is a direct sum of the minimal basic ideals [8].

Based on [1, Lemma 3.4 and Proposition 3.5], if E is a finite acyclic graph and $\{v_1, \ldots, v_m\}$ is a set of sinks, then $I_{v_i} = \sum\{k\alpha\beta^* \mid k \in K; \alpha, \beta \in path(E); r(\alpha) = r(\beta) = v_i\}$ is an ideal in $L_K(E)$, for every $1 \le i \le m$ and $L_K(E) = \oplus_{i=1}^{m} I_{v_i}$, in which $I_{v_i} \cong M_{n(v_i)}(K)$ and $n(v_i)$ is the number of real paths in $path(E)$ ending in vertex v_i. Because $M_{n(v_i)}(K)$ is a simple ring, $L_K(E)$ is a direct sum of minimal ideals I_{v_i}. In other words, $L_K(E)$ is a semisimple algebra if the graph E is finite and acyclic.

If the field K is generalized by commutative unital ring R, then we could show analogously that $I_{v_i} = \sum\{c\alpha\beta^* \mid c \in R; \alpha, \beta \in E^*; r(\alpha) = r(\beta) = v_i\}$ $\cong M_{n(v_i)}(R)$ is an ideal in $L_R(E)$, for every sink $v_i, 1 \le i \le m$ and $L_R(E) = \oplus_{i=1}^{m} I_{v_i}$. If we could prove that I_{v_i} is a minimal basic ideal, then Leavitt path algebra $L_R(E)$, where E is the finite acyclic graph, is basically semisimple. The minimal basic ideal can be created by minimal saturated hereditary subset of E^0. That is one of the focuses in this paper. We also show the necessary and sufficient conditions for $L_R(E)$ as semisimple.

2. On Minimal Saturated Hereditary Subset

Characterization of the Leavitt path algebra $L_R(E)$ can be determined by its ideals, particularly, the basic ideals. A saturated hereditary subset of E^0 is important in constructing the ideals of the Leavitt path algebra, especially the set of sinks.

Definition 2.1 [7]**.** A subset $H \subseteq E^0$ is hereditary if $(\forall e \in E^1)$ $(s(e) \in H \Rightarrow r(e) \in H)$. It is saturated if $(\forall v \in E^0,\ s^{-1}(v) \neq \emptyset)$,

$(r(s^{-1}(v)) \subset H \Rightarrow v \in H)$. Saturation of hereditary subset X, denoted by $\overline{X}$ is the smallest saturated hereditary subset of E^0 containing X.

Intersection of saturated hereditary subsets of E^0 is also saturated hereditary. However, the union of saturated hereditary subsets need not be necessarily saturated. Note the figure of graph G consisting of $G^0 = \{u, v, w, x\}$ and $G^1 = \{e, f, g\}$.

$$u \bullet \xleftarrow{\ e\ } \bullet_v \xrightarrow{\ f\ } \bullet_w \xrightarrow{\ g\ } \bullet_x$$

Figure 1

The graph G is finite and acyclic with sink u and x. Both subsets $\{u\}$ and $\{x\}$ are hereditary with saturation $\overline{\{u\}} = \{u\}$ and $\overline{\{x\}} = \{w, x\}$. In other words, both $\{u\}$ and $\{w, x\}$ are saturated hereditary. However, $\{u\} \bigcup \{w, x\} = \{u, w, x\}$ is hereditary but not saturated.

Lemma 2.2 [7]**.** *Given a graph* $E = (E^0, E^1, r, s)$ *and a commutative unital ring R. If I is an ideal of* $L_R(E)$, *then* $H = \{v \in E^0 : v \in I\}$ *is a saturated hereditary subset.*

The lemma above states that $I \bigcap E^0$ is saturated hereditary for every ideal $I \subseteq L_R(E)$. A saturated hereditary subset of E^0 constructs a graded ideal, as the proposition below asserts.

Proposition 2.3 [7]**.** *Given graph* $E = (E^0, E^1, r, s)$ *and a commutative unital ring R. If* $H \subset E^0$ *is saturated hereditary and* $I_H = \sum \{c\alpha\beta^* : c \in R, \alpha, \beta \in path(E), r(\alpha) = r(\beta) \in H\}$ *is a graded ideal generated by H, then* I_H *is a basic ideal and* $\{v \in E^0 : v \in I_H\} = H$.

Proposition 2.3 states that $E^0 \cap I_H = H$. The hereditary (not necessarily saturated) subset $X \subset E^0$ can generate graded (two-sided) ideal that equal to those generated by the saturation, $\overline{X}$.

Lemma 2.4 [7]**.** *If* $X \subset E^0$ *is hereditary and* I_X *is graded* (*two-sided*) *ideal generated by X, then* $I_X = I_{\overline{X}}$, *where* $\overline{X}$ *is a saturation of X* (*the smallest saturated hereditary subset containing X*).

A non-empty saturated hereditary subset $H \subset E^0$ is a *minimal saturated hereditary* subset if it does not contain any non-empty saturated hereditary subsets except itself. The minimal saturated hereditary subset constructs a minimal basic ideal.

Proposition 2.5. *Given graph* $E = (E^0, E^1, r, s)$ *and a commutative unital ring R. The non-empty saturated hereditary subset* $H \subset E^0$ *is minimal if and only if the basic ideal generated by H,* I_H *is minimal in* $L_R(E)$.

Proof. Based on Proposition 2.3, I_H is a basic ideal. Suppose that the basic ideal I_H is not minimal. Then there is basic ideal $J \subset I_H, J \neq \{0\}$. By Lemma 2.2, $H' = \{v \in E^0 : v \in J\} = E^0 \cap J$ is saturated hereditary. Because $J \subset I_H$, $H' = E^0 \cap J \subset E^0 \cap I_H = H$. This is a contradiction to the fact that H is minimal saturated hereditary. Therefore, I_H is a minimal basic ideal. Conversely, let I_H be a minimal basic ideal. Suppose that the non-empty saturated hereditary subset $H = E^0 \cap I_H$ is not minimal. Then there is a non-empty saturated hereditary subset $H' \subset H$ such that $I_{H'} = \sum\{c\alpha\beta^* : c \in R, \alpha, \beta \in path(E),$ and $r(\alpha) = r(\beta) \in H' \subset H\} \subset I_H$ is a nonzero basic ideal because of Proposition 2.3. It means that a basic ideal I_H is not minimal. It is a contradiction. Therefore, H is minimal. □

3. On Basically Semisimple and Semisimple Leavitt Path Algebra

The graph discussed in this section is a finite graph, that is, the graph is row-finite and E^0 is finite. Graph E is a row-finite graph if $s^{-1}(v)$ is a finite set for every $v \in E^0$. We determine ideals of Leavitt path algebra which are isomorphic to the direct sums of these ideals. However, some definitions and properties associated with the formation of the ideals of the Leavitt path algebra is presented beforehand.

Definition 3.1 [1]**.** For a given graph $E = (E^0, E^1, s, r)$, $path(E)$ is the set of all paths in E. Range index of $v \in E^0$ denoted by $n(v) = \# \{\alpha \in path(E) : r(\alpha) = v\}$, is defined as the cardinality (the number of elements) of the set of real edges ending at v.

Singleton of a sink is a hereditary (not necessarily saturated) subset of E^0. Based on Lemma 2.4, a sink $v \in E^0$ can construct the ideal I_v. If the graph E is finite and acyclic, then I_v is isomorphic to the set of matrices with finite size $n(v)$. The following lemma is a minor generalization of a well-known result in [5, Corollary 2.2], in terms of that I_v is not only an ideal but also a minimal basic ideal.

Lemma 3.2. *Given a finite acyclic graph E, a sink $v \in E^0$ and a commutative unital R. Then $I_v = \sum \{c\alpha\beta^* \mid \alpha, \beta \in E^*, r(\alpha) = v = r(\beta), c \in R\}$ is a minimal basic ideal in $L_R(E)$ and $I_v \cong M_{n(v)}(R)$.*

Proof. First, we show that $I_v = \sum \{c\alpha\beta^* \mid \alpha, \beta \in E^*, r(\alpha) = v = r(\beta), c \in R\}$ is an ideal of $L_R(E)$ and $I_v \cong M_{n(v)}(R)$. This is analogous to [1, Lemma 3.4] by replacing the field K with the ring R. Let $\alpha\beta^* \in I_v$ and let $\gamma\delta^* = e_{i1} \cdots e_{in} e_{j1}^* \cdots e_{jm}^* \in L_R(E)$, be a nonzero monomial. Then $r(\alpha) = v = r(\beta)$. If $\gamma\delta^*\alpha\beta^* = 0$, then $\gamma\delta^*\alpha\beta^* \in I_v$. Therefore, if $\gamma\delta^*\alpha\beta^* \neq 0$, then by equation (1), there are $p, q \in path(E)$ such that $\delta = \alpha q$ or $\alpha = \delta p (CK1)$. Because $\delta = \alpha q$, $\delta \neq 0$ and $r(\alpha) = v$ is a sink, $s(q) = v$ is not possible (impossible $\deg(q) \geq 1$). Thus, $\gamma\delta^*\alpha\beta^* = \gamma g^* \alpha^* \alpha\beta^*$ $= \gamma g^* v \beta^* = 0$, since $\alpha^*\alpha = r(\alpha) = v (CK1)$ and $s(q) \neq v$, implies $g^* v = 0$.

Hence, we use $\alpha = \delta p.$ We know that $r(\alpha) = v,$ then $r(\delta p) = r(p) = v,$ so that

$$\begin{aligned}\gamma\delta^*\alpha\beta^* &= \gamma\delta^*\delta p\beta^* \quad \alpha = \delta p\\ &= \gamma r(\delta)\, p\beta^* \quad \delta^*\delta = r(\delta)\\ &= \gamma s(p)\, p\beta^* \quad r(\delta) = s(p)\\ &= (\gamma p)\beta^* \in I_v \quad r(\beta) = v = r(p) = r(\gamma p).\end{aligned}$$

Hence, I_v is left ideal of $L_R(E)$. We can prove analogously that I_v is the right ideal of $L_R(E)$. Furthermore, we show that $I_v \cong M_{n(v)}(R)$. Because E is finite and acyclic, $n(v) = k < \infty$ and we have $\{p_i \in path(E) : r(p_i) = v,\ i = 1, 2, ..., k\}$ such that the basic ideal I_v is in the form of $I_v = \sum\{cp_i p_j^* | i, j = 1, 2, ..., k; c \in R\}$. Choose $j \neq t$. Suppose that $(p_i p_j^*)(p_t p_u^*) \neq 0$. By equation (1) we have $p_t = p_j q$ or $p_j = p_t p$, for some $p, q \in path(E)$. The first possibility is $p_t = p_j q$. Because of $r(p_t) = v = r(p_j q) = r(q)$ and $j \neq t$, $\deg(q) > 0$. On the other hand, $r(p_j) = v$, and hence $s(q) = r(p_j) = v$, a contradiction to a sink v. The second possibility is $p_j = p_t p$ or $p_j^* = p^* p_t^*$. Then we have $(p_i p_j^*)(p_t p_u^*) = p_i p^* p_t^* p_t p_u^* = p_i p^* v p_u^* \neq 0$ and $p^* v \neq 0$. It means that $s(p) = r'(p^*) = v$ is a contradiction to sink v. Hence, if $j \neq t$, then $(p_i p_j^*)(p_t p_u^*) = 0$. In other words, it should be $j = t$ and we find $(p_i p_j^*)(p_j p_u^*) = p_i v p_u^* = p_i p_u^*$. Therefore, the ideal

$$\begin{aligned}I_v &= \sum\{cp_i p_j^* | i,\ j = 1, 2, ..., k;\ c \in R\}\\ &\cong \sum\{c(e_{ij}) | i,\ j = 1, 2, ..., k;\ c \in R\} = M_k(R),\end{aligned}$$

with $p_i p_j^* \to (e_{ij})$ which is the matrix unit. We obtain $I_v \cong M_k(R)$ $= M_{n(v)}(R)$. Second, we would show that I_v is a minimal basic ideal. Because a sink $v \in E^0$, $\{v\}$ is a hereditary subset of E^0. Based on Lemma 2.4, there is a saturation $H = \overline{\{v\}} \subset E^0$, that is, a minimal saturated hereditary containing v such that $I_H = I_v$. Since I_H is an ideal generated by saturated hereditary H, by Proposition 2.3, $I_H = I_v$ is a basic ideal. In addition, since H is minimal saturated hereditary, based on Proposition 2.5, $I_H = I_v$ is minimal basic ideal. □

According to [7], if all ideals of Leavitt path algebra $L_K(E)$ over a field are basic, then all minimal ideals of $L_K(E)$ are also minimal basic ideals. However, not all minimal ideals of the Leavitt path algebra $L_R(E)$ are minimal basic ideals and conversely, not all minimal basic ideals of $L_R(E)$ are minimal ideals.

Example 3.3. Consider the graph in Figure 1 above and given the commutative unital $R = \mathbb{Z}_4$. Based on Lemma 3.2, because $n(x) = \#\{x, g, fg\} = 3$; $n(u) = \#\{u, e\} = 2$, we have minimal basic ideals in $L_{\mathbb{Z}_4}(G) : I_x \cong \begin{pmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 & \mathbb{Z}_4 \\ \mathbb{Z}_4 & \mathbb{Z}_4 & \mathbb{Z}_4 \\ \mathbb{Z}_4 & \mathbb{Z}_4 & \mathbb{Z}_4 \end{pmatrix}$ and $I_u \cong \begin{pmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 \\ \mathbb{Z}_4 & \mathbb{Z}_4 \end{pmatrix}$. However, both I_x and I_u are not minimal ideals. It is because there are nonzero ideals $J_{2x} \subsetneq I_x$, $J_{2u} \subsetneq I_u$, where

$$J_{2x} = \sum \{c'\alpha\beta^* \mid \alpha, \beta \in path(G), r(\alpha) = x = r(\beta), c' \in 2\mathbb{Z}_4\}$$

$$\cong \begin{pmatrix} 2\mathbb{Z}_4 & 2\mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 2\mathbb{Z}_4 & 2\mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 2\mathbb{Z}_4 & 2\mathbb{Z}_4 & 2\mathbb{Z}_4 \end{pmatrix},$$

and

$$J_{2u} = \sum \{k'\alpha\beta^* \mid \alpha, \beta \in path(G), r(\alpha) = u = r(\beta), k' \in 2\mathbb{Z}_4\}$$

$$\cong \begin{pmatrix} 2\mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 2\mathbb{Z}_4 & 2\mathbb{Z}_4 \end{pmatrix}.$$

Because $2x \in J_{2x}$, $x \notin J_{2x}$ and $2u \in J_{2u}$, $u \notin J_{2u}$, both J_{2x}, J_{2u} are not basic ideals but minimal ideals.

Ideal I_v generated by a sink $v \in E^0$ is not necessarily a minimal ideal but certainly a minimal basic ideal in $L_R(E)$. Furthermore, we are inspired by [5, Corollary 2.3] to investigate that $L_R(E)$ on the finite acyclic graph is a direct sum of minimal basic ideals generated by the sinks.

Theorem 3.4. *If given the finite acyclic graph E and the set $\{v_i \in E^0 : v_i \text{sink}, i = 1, 2, ..., t\}$, then $L_R(E) = \oplus_{i=1}^t I_{v_i} \cong \oplus_{i=1}^t M_{n(v_i)}(R)$.*

Proof. Based on Lemma 3.2, $I_{v_i} = \sum \{k\alpha\beta^* \mid \alpha, \beta \in E^*, r(\alpha) = v_i = r(\beta)\}$ is a graded basic ideal in $L_R(E)$ and $I_{v_i} \cong M_{n(v_i)}(R)$, for every $i = 1, 2, ..., t$. Furthermore, we prove that $L_R(E) = \sum_{i=1}^t I_{v_i}$. Take arbitrary $\alpha\beta^* \in L_R(E)$; $\alpha\beta^* \neq 0$, $\alpha, \beta \in path(E)$. If $r(\alpha) = v_i = r(\beta)$ for some sinks v_i, then $\alpha\beta^* \in I_{v_i}$. If $r(\alpha) \neq v_i$ for every sink v_i, then $r(\alpha)$ is not a sink. It means that there is an edge $e \in E^1$ such that $r(\alpha) = s(e) = v$. Because $\alpha\beta^* \in L_R(E)$ and by CK2 condition, we have

$$\alpha\beta^* = \alpha v\beta^* = \alpha \sum_{\substack{e \in E', \\ s(e)=v=r(\alpha)}} ee^*\beta^* = \sum_{\substack{e \in E', \\ s(e)=v=r(\alpha)}} \alpha ee^*\beta^* = \sum_{\substack{e \in E', \\ s(e)=v=r(\alpha)}} (\alpha e)(\beta e)^*.$$

If $r(ae) = v_i = r(\beta e)$ for some sinks v_i, then $\alpha\beta^* \in I_{v_i}$. However, if $r(ae) \neq v_i$ for every sink v_i, then the process is repeated to obtain a sink v_i

such that $\alpha\beta^* \in I_{v_i}$ caused by: $\alpha\beta^* = \sum_{\gamma\in Path(E)}(\alpha\gamma)(\beta\gamma)^*$ with $r(\alpha\gamma) = v_i$ $= r(\beta\gamma)$ for some sinks v_i. Thus, $L_R(E) = \sum_{i=1}^{t} I_{v_i}$. Finally, take any $\alpha\beta^* \in I_{v_i}$ and $\gamma\delta^* \in I_{v_j}$ with $i \neq j$. Since v_i and v_j are sinks, there is no path $\beta\gamma' = \gamma$ or $\gamma\beta' = \beta$ such that $(\alpha\beta^*)(\gamma\delta^*) \neq 0$. In other words, $(\alpha\beta^*)(\gamma\delta^*) = 0$ for every $i \neq j$, $i, j = 1, 2, \ldots, t$. It should be $I_{v_i} I_{v_j} = 0$ for every $i \neq j, i, j = 1, 2, \ldots, t$. Therefore, if $\alpha\beta^* \in I_{v_i}$ and $\gamma\delta^* \in I_{v_j}$ with $i \neq j$, then $\beta^* \neq \gamma\delta^*$. However, $0 \in I_{v_i}$ for every $i = 1, 2, \ldots, t$, because $0 \in R$. Then $I_{v_i} \cap I_{v_j} = \{0\}$, for every $v_i \neq v_j$. We find $L_R(E) = \oplus_{i=1}^{t} I_{v_i}$ and by Lemma 3.2, $I_{v_i} \cong M_{n(v_i)}(R)$ for every sink v_i. Then $L_R(E) \cong \oplus_{i=1}^{t} M_{n(v_i)}(R)$. □

Based on [8, Definition 3.4], the Leavitt path algebra is basically semisimple if it is a direct sum of minimal basic ideals. Theorem 3.4 above has the following corollary.

Corollary 3.5. *Any Leavitt path algebra over a commutative unital ring on a finite acyclic graph is basically semisimple, in which* $L_R(E) = \oplus_{v\,\text{sink}} I_v \cong \oplus_{v\,\text{sink}} M_{n(v)}(R)$.

Example 3.6. Consider again Example 3.3, then the Leavitt path algebra over $\mathbb{Z}_4$ on the graph G,

$$L_{\mathbb{Z}_4}(G) = I_u \oplus I_x \cong \begin{pmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 \\ \mathbb{Z}_4 & \mathbb{Z}_4 \end{pmatrix} \oplus \begin{pmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 & \mathbb{Z}_4 \\ \mathbb{Z}_4 & \mathbb{Z}_4 & \mathbb{Z}_4 \\ \mathbb{Z}_4 & \mathbb{Z}_4 & \mathbb{Z}_4 \end{pmatrix}$$

$$\cong \left(\begin{pmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 \\ \mathbb{Z}_4 & \mathbb{Z}_4 \end{pmatrix}, \begin{pmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 & \mathbb{Z}_4 \\ \mathbb{Z}_4 & \mathbb{Z}_4 & \mathbb{Z}_4 \\ \mathbb{Z}_4 & \mathbb{Z}_4 & \mathbb{Z}_4 \end{pmatrix} \right)$$

is basically semisimple. However, $L_{\mathbb{Z}_4}(G)$ is not semisimple. If ring $\mathbb{Z}_4$ is replaced by semisimple ring $\mathbb{Z}_6$, then we find

$$L_{\mathbb{Z}_6}(G) = I_u \oplus I_x \cong \left(\begin{pmatrix} \mathbb{Z}_6 & \mathbb{Z}_6 \\ \mathbb{Z}_6 & \mathbb{Z}_6 \end{pmatrix}, \begin{pmatrix} \mathbb{Z}_6 & \mathbb{Z}_6 & \mathbb{Z}_6 \\ \mathbb{Z}_6 & \mathbb{Z}_6 & \mathbb{Z}_6 \\ \mathbb{Z}_6 & \mathbb{Z}_6 & \mathbb{Z}_6 \end{pmatrix} \right)$$

a basically semisimple Leavitt path algebra that is semisimple, because

$$L_{\mathbb{Z}_6}(G) = \underbrace{I_u \oplus I_x}_{\substack{\text{a direct sum of} \\ \text{minimal basic ideals}}} = \underbrace{I_{2u} \oplus I_{3u} \oplus I_{2x} \oplus I_{3x}}_{\substack{\text{a direct sum of} \\ \text{minimal ideals} \\ \text{that not basic ideals}}},$$

with

$$I_u = I_{2u} \oplus I_{3u} \text{ and } I_x = I_{2x} \oplus I_{3x},$$

in which

$$I_{2u} \cong \begin{pmatrix} 2\mathbb{Z}_6 & 2\mathbb{Z}_6 \\ 2\mathbb{Z}_6 & 2\mathbb{Z}_6 \end{pmatrix}, \; I_{3u} \cong \begin{pmatrix} 3\mathbb{Z}_6 & 3\mathbb{Z}_6 \\ 3\mathbb{Z}_6 & 3\mathbb{Z}_6 \end{pmatrix},$$

$$I_{2x} \cong \begin{pmatrix} 2\mathbb{Z}_6 & 2\mathbb{Z}_6 & 2\mathbb{Z}_6 \\ 2\mathbb{Z}_6 & 2\mathbb{Z}_6 & 2\mathbb{Z}_6 \\ 2\mathbb{Z}_6 & 2\mathbb{Z}_6 & 2\mathbb{Z}_6 \end{pmatrix}, \text{ and } I_{3x} \cong \begin{pmatrix} 3\mathbb{Z}_6 & 3\mathbb{Z}_6 & 3\mathbb{Z}_6 \\ 3\mathbb{Z}_6 & 3\mathbb{Z}_6 & 3\mathbb{Z}_6 \\ 3\mathbb{Z}_6 & 3\mathbb{Z}_6 & 3\mathbb{Z}_6 \end{pmatrix}.$$

The Leavitt path algebra on a finite acyclic graph is always basically semisimple but not necessarily semisimple. Based on [6, Corollary 18.6], $M_n(R)$ is semisimple if and only if R is semisimple.

Theorem 3.7. *The Leavitt path algebra $L_R(E)$ on a finite acyclic graph is semisimple if and only if the commutative unital ring R is also semisimple.*

Proof. Let $L_R(E)$ be a semisimple R-algebra. Then, it could be viewed as a semisimple R-module. Based on Lemma 3.2 and Corollary 3.5, every sink $I_v \cong M_{n(v)}(R)$ is R-submodule of $L_R(E) = \oplus_{v\,\text{sink}} I_v \cong \oplus_{v\,\text{sink}} M_{n(v)}(R)$. In [2, Corollary 1.24] is stated that sums and submodules of semisimple modules are semisimple. Thus, $I_v \cong M_{n(v)}(R)$ is semisimple. By [6, Corollary 18.6], R is semisimple. The converse follows from [6, Corollary 18.6] and [2, Corollary 1.24].

4. Conclusion

Singleton of a sink is a hereditary subset but not necessarily saturated. Every sink in a finite acyclic graph E will generate minimal basic ideal I_v of Leavitt path algebras $L_R(E)$ over commutative unital ring R on the graph, in which $I_v \cong M_{n(v)}(R)$ and $n(v)$ is range index of v. In addition, $L_R(E) = \oplus_{v\,\text{sink}}\ I_v \cong \oplus_{v\,\text{sink}} M_{n(v)}(R)$ are direct sums of minimal basic ideals generated by the sinks, and hence is a basically semisimple algebra.

The minimal basic ideal I_v is not necessarily minimal in $L_R(E)$. This means that the basically semisimple $L_R(E)$ is not necessarily semisimple. Every $L_R(E)$ on a finite acyclic graph is semisimple if and only if the commutative unital R is semisimple.

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