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Disturbance attenuation problem using a differential game approach for feedback linear quadratic descriptor systems

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This paper studies the $\mathcal{H}_\infty$ disturbance attenuation problem for index one descriptor systems using the theory of differential games. To solve this disturbance attenuation problem the problem is converted into a reduced ordinary zero-sum game. Within a linear quadratic setting the problem is solved for feedback information structure.

Keywords: disturbance attenuation problem, zero-sum linear quadratic differential game, descriptor systems, feedback information structure.

1. Introduction

The disturbance attenuation problem is an important issue in many practical applications since all practical control systems are subject to disturbances. Any form of disturbance is always detrimental to systems. Therefore, one of the biggest problems faced by a control designer is to design a controller that is capable to overcome various form of disturbances that might arise in the system [11]. One of the techniques to solve a disturbance attenuation problem uses a differential game approach [1], [2], and [14]. In this disturbance attenuation framework the control designer is the minimizing agent. They fight against disturbances or uncertainties which are represented by the maximizing agent.

The main problem discussed in this paper is how to solve the disturbance attenuation problem by designing a robust optimal control, using a differential game approach, for descriptor systems that have index one, assuming a feedback information framework. [1] used this approach to find robust controllers for regular systems (nonsingular systems). In [21] the feedback zero-sum linear quadratic soft-constrained descriptor differential game for index one systems was solved by transforming it into a regular differential game. Following this same procedure, the robust control design problem

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for descriptor systems can also be translated into a regular differential game. Hence, by merging results obtained in [1] and [21], we will solve the robust optimal control problem for descriptor systems here. Further, [20] solved this problem for an open-loop information framework.

Compared to the frequency-domain formulation the addressed technique, which is a time-domain approach, has the advantage that it allows one to formulate also finite planning horizon and time-varying versions of the problem, and thus also study the transient behavior [1], [18]. Moreover, the addressed technique seems to be the simplest and the most intuitive one, since after all the original robust optimal control problem is a minimax optimization control problem and hence a zero-sum game.

As already noted above, in this paper we consider the disturbance attenuation problem under the assumption that the dynamics of the underlying process are described by a descriptor system that has index one. Descriptor systems model static as well as dynamic constraints of a real plant using sets of coupled differential and algebraic equations. Applications of descriptor systems can be found in chemical processes [13], circuit systems [23, 24], economic systems [16], large-scale interconnected systems [17, 28], mechanical engineering systems [9], power systems [27], and robotics [19].

The organization of this paper is as follows. Section 2 introduces the precise problem formulation of designing a disturbance attenuation problem by designing a robust optimal control using the differential game approach and translates the problem for a descriptor system into a regular differential game. Section 3 recalls from [21] the feedback zero-sum linear quadratic soft-constrained descriptor differential game for both a finite planning horizon and an infinite planning horizon. Section 4 presents the main result for the problem. We discuss the solution for the disturbance attenuation problem for both a finite and infinite planning horizon. At last, section 6 will conclude.

2. Problem statement

We consider in this paper the dynamical system

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + B_1u(t) + B_2w(t), x(0) = x_0 \\ y(t) &= C_1x(t) + D_{12}w(t) \\ z(t) &= C_2x(t) + D_{21}u(t) \end{aligned} \tag{1}$$

where $E, A \in \mathbb{R}^{(n+r)\times(n+r)}$, $rank(E) = n$, $B_i \in \mathbb{R}^{(n+r)\times m_i}$, $C_1 \in \mathbb{R}^{p\times(n+r)}$, $C_2 \in \mathbb{R}^{q\times(n+r)}$, $D_{12} \in \mathbb{R}^{p\times m_2}$ and $D_{21} \in \mathbb{R}^{q\times m_1}$. Vector $u \in U_s$ models the action performed by the control designer to control the system, where U_s denotes the set of locally square integrable control functions yielding a stable closed-loop system. Vector $w \in L_2(0,\infty)$ represents disturbances and uncertainties arising in the system, where $L_2(0,\infty)$ denotes the set of all measurable Lebesgue square integrable functions on $(0,\infty)$. Vector $y(t)$ is the measured output and $z(t)$ is the controlled output. In the general context, the control $u(t)$ is assumed to be a linear mapping from a subset of the measured outputs $y(s), s \leq t$.

The consistent initial state, x_0, is known (and is zero in the disturbance attenuation problem). Moreover, the controller is allowed to have perfect access to the system state (perfect-state measurements). System (1) is called regular if and only if $\det(\lambda E - A) \neq 0$. From [7] we recall the so-called Weierstrass' canonical form.

Theorem 1 *Assume that* (1) *is regular. Then, there exist nonsingular matrices X and Y such that*

$$Y^T E X = \begin{bmatrix} I_n & 0 \\ 0 & N \end{bmatrix} \text{ and } Y^T A X = \begin{bmatrix} A_1 & 0 \\ 0 & I_r \end{bmatrix}. \tag{2}$$

Here A_1 is a matrix in Jordan form whose elements are the finite eigenvalues, $I_k \in \mathbb{R}^{k\times k}$ is the identity matrix, and $N \in \mathbb{R}^{r\times r}$ is a nilpotent matrix that is also in Jordan form. A_1 and N are unique up to a permutation of Jordan blocks.

System (1) is said to be of index one[1] if $N = 0$ (its degree of nilpotency is one). Next, recall from, e.g., [8] that for all initial states in (1) there exists a smooth control that generates a smooth state trajectory if and only if (1) is impulse controllable[2]. Further, all impulsive modes of (1) can be transformed then into finite dynamic modes using static state feedback control. Since we do not want to consider impulsive control actions in this paper we restrict our attention to impulse controllable systems. As we consider the state feedback control problem, this motivates why we may assume for our problem, without loss of generality, that the system (1) has index one. The above discussion motivates then the next assumptions.

Assumption 1. *We use the following assumptions throughout this paper:*

1. *matrix E is singular*
2. $\det(\lambda E - A) \neq 0$
3. *system* (1) *is impulse controllable.*

Assume the transfer function for the system (1) is

$$G_{zw} := \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}. \tag{3}$$

Let $\hat{K}$ denote the transfer function of the controller. Each controller $\hat{K}$ induces a linear map $z = G_{zw} w$, where the transfer function G_{zw} is given by the linear fractional transformation (LFT),

$$G_{zw} = G_{11} + G_{12}\left(I - \hat{K}\cdot G_{22}\right)^{-1}\left(\hat{K}\cdot G_{21}\right).$$

[1] This equivalent (see e.g. [12]) to the assumption that $rank([E \quad AW]) = n + r$, where the image of matrix W equals the null space of E.

[2] System (1) is impulse controllable (see e.g. [12]) if and only if $rank([E \quad AW \quad B]) = n + r$.

The design objective is to find a control that will keep the output (z) small, regardless of unpredictable disturbances (w). In mathematical term: Given a nonnegative number γ, find, if it exists, a controller $\hat{K}$ such that

$$\frac{\| z \|}{\| w \|} \leq \gamma .$$

Or, in terms of the above LFT, find a controller such that the norm of the linear operator G_{zw} is smaller than γ. In case by the controller induced linear system is stable, the induced linear operator norm of $G_{zw} = \sup_{w \in L_2(0,\infty)} \frac{\|G_u(w)\|}{\|w\|}$ exists and equals the $\mathcal{H}_\infty$ norm of G_{zw} (see, e.g., Proposition 1.1 in [1]). Here $G_u(w)$ denotes a bounded causal linear operator from w to z, i.e., $z = G_u(w)$. Then, the design problem can be reformulated into the following optimization problem. Find:

$$\inf_{u \in U_s} \|G_{zw}\|_\infty = \inf_{u \in U_s} \sup_{w \in L_2(0,\infty)} \frac{\|G_u(w)\|}{\|w\|}. \tag{4}$$

Denote this infimum by γ^*. Unfortunately, this infimum cannot be realized by choosing a specific stabilizing controller. Therefore, usually, the addressed problem is restated as finding an admissible optimal controller $u^* \in U_s$, for a given attenuation level γ (which must of course be larger than γ^*), such that $\|G_{zw}\|_\infty < \gamma$.

To motivate the introduction of the soft-constrained control problem below, note that the right side in equation (4) defines an upper value for the game defined by dy namical system (1) with objective function $\frac{\|G_u(w)\|}{\|w\|}$. Assume (see [1]) that there exists a control policy $u^* \in U_s$, and a corresponding γ^*, that satisfies (4). Then (4) can be equivalently expressed as

1. there exist $u^* \in U_s$ and corresponding $\tilde{\gamma}^*$ such that $\|G_{u^*}(w)\|^2 \leq \tilde{\gamma}^* \|w\|^2$, for all $w \in L_2(0,\infty)$, where $\tilde{\gamma} = \sqrt{\gamma}$, and

2. there exist no other $u \in U_s$ (say, $\hat{u}$), and a corresponding $\hat{\gamma} < \gamma^*$, such that $\|G_{\hat{u}}(w)\|^2 \leq \hat{\gamma}^* \|w\|^2$, for all $w \in L_2(0,\infty)$.

Now, consider a parameterized family of cost functions (in $\gamma \geq 0$),

$$J_\gamma(u,w) := \|G_u(w)\|^2 - \gamma \|w\|^2 . \tag{5}$$

Then, the statement 1. and 2. can be restated as finding the smallest $\gamma \geq 0$ under which the upper value of the game defined by (1) with objective function (5) is bounded above by zero, and a controller that achieves this upper value. Or, finding the minimal γ for which

$$\inf_{u\in U_s} \sup_{w\in L_2(0,\infty)} J_\gamma(u,w).$$

exist. Next, defining the following norm

$$\|G_u(w)\|^2 := \int_0^\infty z^T(t)z(t)\,dt$$

$$\|w\|^2 := \int_0^\infty (Ww)^T(t)Ww(t)\,dt$$

and adding the assumption

$$C_2^T D_{21} = 0, \tag{6}$$

for the dynamical system (1) then the cost function (5) can be represented in a quadratic form,

$$J_\gamma(u,w) = \int_0^\infty \left[x^T(t)\bar{Q}x(t) + u^T(t)\bar{R}_1 u(t) - \gamma w^T(t)\bar{R}_2 w(t)\right]dt. \tag{7}$$

Here $\bar{Q} := C_2^T C_2$, $\bar{R}_1 := D_{21}^T D_{21}$, and $\bar{R}_2 := W^T W$. This converts the robust control design problem into a linear quadratic zero-sum game defined by system (1) where the cost function for the first player (control designer) is (7) and for the second player (nature, disturbances and uncertainties) $-J_\gamma(u,w)$. The game defined by (1,7) is called the (zero-sum) soft-constrained differential game. This terminology is used to capture the feature that in this game there is no hard bound with respect to w [1]. In this closed-loop game framework we assume that the controls given by both players are in linear feedback control form defined by

$$u(t) = F_1(t)x(t) \in U_s \ \ and\ \ w(t) = F_2(t)x(t) \in L_2(0,\infty) \tag{8}$$

where $F_i(t), i = 1,2$ is a piecewise continuous function and $u(t), w(t)$ depend only on the current state of the systems and time. So, for a fixed γ, the robust control design problem reduces to finding the optimal controller $u^* \in U_s$ that satisfies

$$\inf_{u\in U_s} \sup_{w\in L_2(0,\infty)} \int_0^\infty \left[x^T(t)\bar{Q}x(t) + u^T(t)\bar{R}_1 u(t) - \gamma w^T(t)\bar{R}_2 w(t)\right]dt, \tag{9}$$

subject to dynamical equation (1).

Following the procedure in [21], yields that (u^*, w^*) is a Feedback Saddle Point (FBSP) solution for the zero-sum differential game (1) with cost function (7) if and only if $(\tilde{F}_1^*, \tilde{F}_2^*)$ is a FBSP solution f or the zero-sum differential game defined by

$$\begin{aligned}\dot{x}_1(t) &= \left(A_1 + \begin{bmatrix} B_{11} & B_{21} \end{bmatrix} \begin{bmatrix} \tilde{F}_1 \\ \tilde{F}_2 \end{bmatrix}\right) x_1(t),\ x_1(0) = \begin{bmatrix} I_n & 0 \end{bmatrix} X^{-1} x_0 \\ y(t) &= \bar{C}_{11} x_1(t) + \bar{C}_{12} x_2(t) + D_{12} w(t) \\ z(t) &= \bar{C}_{21} x_1(t) + \bar{C}_{22} x_2(t) + D_{21} u(t),\end{aligned} \tag{10}$$

where $C_1 X := \bar{C}_1 = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} \end{bmatrix}$, $C_2 X := \bar{C}_2 = \begin{bmatrix} \bar{C}_{21} & \bar{C}_{22} \end{bmatrix}$, $B_{i1} = \begin{bmatrix} I & 0 \end{bmatrix} Y^T B_i$,

$B_{i2} = \begin{bmatrix} 0 & I \end{bmatrix} Y^T B_i$, $\tilde{F}_i = F_i X \begin{bmatrix} I \\ H \end{bmatrix}$, $H = -\left(I + \begin{bmatrix} B_{12} & B_{22} \end{bmatrix} F X_2\right)^{-1} \begin{bmatrix} B_{12} & B_{22} \end{bmatrix} F X_1$,

$Y^T = \begin{bmatrix} Y_1^T & Y_2^T \end{bmatrix}$, $X = \begin{bmatrix} X_1 & X_2 \end{bmatrix}$ are nonsingular matrices as defined in (2) and

$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} := X^{-1} x(t)$. The cost function for the control designer is

$$J_\gamma\left(\tilde{F}_1, \tilde{F}_2\right) = \int_0^\infty \left\{ x_1^T(t) \begin{bmatrix} I & \tilde{F}_1^T & \tilde{F}_2^T \end{bmatrix} \tilde{M}_\gamma \begin{bmatrix} I \\ \tilde{F}_1 \\ \tilde{F}_2 \end{bmatrix} x_1(t) \right\} dt \tag{11}$$

where

$$\tilde{M}_\gamma = \begin{bmatrix} \tilde{Q} & \tilde{V} & \tilde{W} \\ \tilde{V}^T & \tilde{R}_{11} & \tilde{N} \\ \tilde{W}^T & \tilde{N}^T & \tilde{R}_{22\gamma} \end{bmatrix}, \tag{12}$$

$\tilde{R}_{11} > 0$ and $\tilde{R}_{22\gamma} > 0$. In the Appendix we provided an adequate explanation about the transformation from descriptor differential game (1,7) into a regular reduced differential game (10,11), as well as the spellings of the matrices defined in (12).

3. The feedback zero-sum linear quadratic soft-constrained descriptor differential game

In this section we recall some theorems of feedback zero-sum linear quadratic soft-constrained differential games for index one descriptor systems from [21] that will be used as a framework in designing robust optimal controls. We will characterize the set of FBSP solutions for the game (1,7) using the reduced ordinary differential game described by the dynamical system (10) with the cost function (11). The general index case was studied in [25], where the theory of projector chains is used to decouple algebraic and differential parts of the descriptor system, and then the usual theory of ordinary differential games is applied to derive both necessary and sufficient conditions for the existence of feedback Nash equilibria for linear quadratic differential games. Furthermore, the open-loop version of such a game has been studied in [22], while the hard constrained version for such games can be found in [26] and [30]. We assume that the players act non-cooperatively and the information they have is the present state and the model structure.

In addition to linear feedback control in the form of equation (8), in this section we also restrict the controller in the sense that it must stabilize the system for all consistent initial states. As discussed in [5] we assume that the feedback matrix F belongs to the set

$$\mathcal{F} := \left\{ F = \begin{bmatrix} F_1^T & F_2^T \end{bmatrix}^T \left| \begin{array}{l} all\ finite\ eigenvalues\ of\ (E, A+BF)\ are\ stable \\ and\ (E, A+BF)\ has\ index\ one, \end{array} \right. \right\} \tag{13}$$

where $B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$. We assume that the matrix pairs $(A_1, B_{i1}), i = 1,2$, are stabilizable. So, in principle, each player is capable to stabilize the first part of the transformed system on his own. Furthermore, to ensure that such a stabilizing control for player one is generated naturally through the solution (optimization) process, unstable modes of A_1 will have to be observable through $\tilde{Q}$. So, we make the assumption that the pair $\left(A_1, \tilde{Q}^{\frac{1}{2}} \right)$ is detectable [1]. To find the FBSP solution the following algebraic Riccati equation (ARE) plays an important role

$$A_1^T K + KA_1 + \tilde{Q} - \begin{bmatrix} \tilde{V} + KB_{11} & \tilde{W} + KB_{21} \end{bmatrix} \tilde{G}_\gamma^{-1} \begin{bmatrix} \tilde{V} + KB_{11} & -(\tilde{W} + KB_{21}) \end{bmatrix}^T = 0. \tag{14}$$

where $\tilde{G}_\gamma = \begin{bmatrix} \tilde{R}_{11} & \tilde{N} \\ -\tilde{N}^T & \tilde{R}_{22\gamma} \end{bmatrix}$. Furthermore, to ensure the ARE above has a solution, we introduce the following nonempty set

$$\Gamma_{CL}^\infty = \left\{ \inf \Gamma_1^\infty, \inf \Gamma_2^\infty \right\},\ \gamma_{CL}^\infty = \max \Gamma_{CL}^\infty, \tag{15}$$

where

$$\Gamma_1^\infty = \left\{ \tilde{\gamma} > 0 \middle| \forall \gamma \geq \tilde{\gamma}, \tilde{R}_{22\gamma} > 0 \right\},$$

$$\Gamma_2^\infty = \left\{ \tilde{\gamma} > 0 \middle| \forall \gamma \geq \tilde{\gamma}, \bar{J}_\gamma := \inf_{F_1} \sup_{F_2} J_\gamma (F_1, F_2) \leq 0 \right\},$$

and $\bar{J}_\gamma$ denotes the upper value of the game. Then, applying Theorem 3 in [5] to Theorem 4.8 in [1] yields the following theorem.

Theorem 2 *Consider the infinite-horizon linear quadratic soft-constrained differential game* (1,7) *with feedback information structure, consistent initial state* $x_0 \in \mathbb{R}^{n+r}$ *is arbitrary and* $Q_\infty = 0$. *Moreover assume that* $\tilde{R}_{11}, \tilde{R}_{22\gamma} > 0$, (6) *holds and the pair* $\left(A_1, \tilde{Q}^{\frac{1}{2}}\right)$ *is detectable. Then:*

1. *For each fixed* t, *the solution to*

$$\dot{K}(t) = -A_1^T K(t) - K(t) A_1 - \tilde{Q} + \left[\tilde{V} + K(t) B_{11} \quad \tilde{W} + K(t) B_{21}\right] \tilde{G}_\gamma^{-1} \times \left[\tilde{V} + K(t) B_{11} \quad -\left(\tilde{W} + K(t) B_{21}\right)\right]^T, \qquad K\left(t_f\right) = \bar{Q}_{t_f}, \tag{16}$$

$K_\gamma\left(t;t_f\right)$, *is non-decreasing in* t_f, *that is if* (16) *has no conjugate point in a given interval* $\left[0, t_f\right]$,

$$K_\gamma\left(t;t_f^{'}\right) - K_\gamma\left(t;t_f^{''}\right) \geq 0,\ t_f > t_f^{'} > t_f^{''} \geq 0.$$

2. *If there exists a nonnegative definite solution of* (14), *there is a minimal solution, denoted* K_γ^+. *This matrix has the property that* $K_\gamma^+ - K_\gamma\left(t;t_f\right) \geq 0$ *for all* $t_f \geq 0$, *where* $K_\gamma\left(t;t_f\right)$ *is the solution of* (16) *with* $Q_{t_f} = 0$. *If* $\left(A_1, \tilde{Q}^{\frac{1}{2}}\right)$ *is observable, then every nonnegative definite solution of* (14) *is positive definite.*

3. *The differential game* (1,7) *has equal upper and lower values if and only if the ARE* (14) *admits a nonnegative definite solution, in which case the common value is*

$$L_\gamma^{t_f *} = x_0^T X^{-T} \left[I \quad 0\right]^T \bar{K}_\gamma^+ \left[I \quad 0\right] X^{-1} x_0.$$

4. *If the upper value is finite for* $\gamma_{CL}^\infty > 0$, *then it is bounded and equals the lower value for all* $\gamma > \gamma_{CL}^\infty$.

5. *If* $K_\gamma^+ \geq 0$ *exists, let* F_i^* *be given by*

$$F_i^* = \tilde{F}_i^* O^+ + Z_i\left(I - OO^+\right) \tag{17}$$

where $Z_i \in \mathbb{R}^{m_i \times (n+r)}$, $O = X\begin{bmatrix} I \\ -B_{12}\tilde{F}_1^* - B_{22}\tilde{F}_2^* \end{bmatrix}$ *and* $\left(\tilde{F}_1^*, \tilde{F}_2^*\right)$ *are given by*

$$\begin{bmatrix} \tilde{F}_1^* \\ \tilde{F}_2^* \end{bmatrix} = -\tilde{G}_\gamma^{-1} \begin{bmatrix} B_{11}^T K_\gamma^+ + \tilde{V}^T \\ -B_{21}^T K_\gamma^+ - \tilde{W}^T \end{bmatrix}. \tag{18}$$

Then, F_1^* is the steady-state feedback controller that attains the finite upper value, in the sense that

$$\sup_{F_2} J_\gamma\left(F_1^*, F_2\right) = L_\gamma^{\infty *}, \tag{19}$$

and F_2^* is the maximizing feedback solution in (11). *The pair $\left(F_1^*, F_2^*\right)$ constitutes a FBSP solution for the game* (1,7).

6. *If the upper value is finite for some $\gamma > \gamma_{CL}^\infty$, then for all $\gamma > \gamma_{CL}^\infty$ the reduced closed-loop system $\dot{x}_1 = A_{1_{CL}} x_1$ and $\dot{x}_1 = \hat{A}_{1_{CL}} x_1$ where*

$$A_{1_{CL}} := A_1 - \begin{bmatrix} B_{11} & B_{21} \end{bmatrix} \tilde{G}_\gamma^{-1} \begin{bmatrix} B_{11}^T K_\gamma^+ + \tilde{V}^T \\ -B_{21}^T K_\gamma^+ - \tilde{W}^T \end{bmatrix} \tag{20}$$

and

$$\hat{A}_{1_{CL}} := A_1 - \begin{bmatrix} B_{11} & 0 \end{bmatrix} \tilde{G}_\gamma^{-1} \begin{bmatrix} B_{11}^T K_\gamma^+ + \tilde{V}^T \\ 0 \end{bmatrix} \tag{21}$$

are both asymptotically stable.

7. *For $\gamma > \gamma_{CL}^\infty$, $K_\gamma^+ \geq 0$ is the unique solution of* (14) *in the class of nonnegative definite matrices which make $A_{1_{CL}}$ stable.*

8. *Since $\left(A_1, B_{i1}\right), i = 1,2$ are stabilizable then the upper value is bounded for $\gamma > \gamma_{CL}^\infty$.*

Proof We can restrict for the proof of most statements to the reduced order system. For part 1–4 see part (i) – (iv) of Theorem 4.8 from [1]. Part 5 follows by using the relationship (38) again, using the results of Theorem 3 from [5]. Finally, part 6–8 result again, from part (vi) – (viii) of Theorem 4.8 in [1]. □

The more general version of Theorem 2 can be obtained by relaxing the detectability condition of $\left(A_1, \tilde{Q}^{\frac{1}{2}}\right)$. By letting $\gamma^{-2} = 0$ in (14) we arrive at the standard algebraic Riccati equation that arises in linear regulator theory for descriptor systems (see also [1] page 142 for the ordinary game)

$$A_1^T \Sigma + \Sigma A_1 - \left(\Sigma Y_1 B_1 + \tilde{V}\right) R_{11}^{-1} \left(B_1^T Y_1^T \Sigma + \tilde{V}^T\right) + Q = 0. \tag{22}$$

Let $\tilde{\Sigma}^+ \geq 0$ denote its maximal solution. We are now in a position to state the following theorem (For a proof we refer to Theorem 4.8’ in [1]).

Theorem 3. *Consider the framework of Theorem 2, but with* $(A_1, B_{i1}), i = 1,2,$ *stabilizable and* $\left(A_1, \tilde{Q}^{\frac{1}{2}}\right)$ *having no unobservable modes on the imaginary axis. Moreover assume that* $\tilde{R}_{11}, \tilde{R}_{22\gamma} > 0$*, and* (6) *holds. Let* Σ^+ *be the unique maximal solution of* (22). *Then:*

1. *There exists a finite* $\gamma_{CL}^{\infty} \geq 0$*, such that for all* $\gamma > \gamma_{CL}^{\infty}$ *the upper value is bounded and for* $\gamma < \gamma_{CL}^{\infty}$ *it is unbounded.*
2. *For all* $\gamma > \gamma_{CL}^{\infty}$*, in the class of all symmetric matrices bounded from below by* Σ^+ *there is a minimal one that solves* (14)*, to be denoted* $\tilde{K}_{\gamma}^{+}$*. This matrix has the additional property that* $\tilde{K}_{\gamma}^{+} - K_{\gamma}(t;t_f) \geq 0$ *for all* $t_f \geq 0$*, where* $K_{\gamma}(t;t_f)$ *is the solution of* (16) *with* $Q_{t_f} = \tilde{\Sigma}^+$ *and* γ *fixed.*
3. *For all* $\gamma > \gamma_{CL}^{\infty}$*, the differential game* (1,7) *has equal upper and lower value. If player one is restricted to stabilizing controllers, the common value is*

$$L_{\gamma}^{\infty *} = x_0^T X^{-T} \begin{bmatrix} I & 0 \end{bmatrix}^T \tilde{K}_{\gamma}^{+} \begin{bmatrix} I & 0 \end{bmatrix} X^{-1} x_0.$$

4. *For all* $\gamma > \gamma_{CL}^{\infty}$*, the steady-state feedback controller* F_i^* *is given by*

$$F_i^* = \tilde{F}_i^* O^+ + Z_i\left(I - OO^+\right), \text{ where } Z_i \in \mathbb{R}^{m_i \times (n+r)}, \quad O = X\begin{bmatrix} I \\ -B_{12}\tilde{F}_1^* - B_{22}\tilde{F}_2^* \end{bmatrix} \text{ and}$$

$\left(\tilde{F}_1^*, \tilde{F}_2^*\right)$ *are given by* (18). F_1^* *attains the finite upper value* $L_{\gamma}^{\infty *}$*, and*

$$\sup_{F_2} J_{\gamma}\left(F_1^*, F_2\right) = L_{\gamma}^{\infty *}.$$

The maximizing feedback solution above is again given by F_2^*.

5. *For all* $\gamma > \gamma_{CL}^{\infty}$*, the two matrices (20) and* (21) *are stable (again with* $\tilde{K}_{\gamma}^{+}$ *taken above).*
6. *For all* $\gamma > \gamma_{CL}^{\infty}$*, the matrix* $\tilde{K}_{\gamma}^{+}$ *defined in Theorem* 2 *is the unique solution of* (14) *in the class of nonnegative definite matrices which make* (20) *stable.*
7. *For* $\gamma < \gamma_{CL}^{\infty}$*, the algebraic Riccati equation* (14) *has no real solution that also makes* (21) *stable.*

4. Disturbance attenuation problem

To eliminate the effect of disturbances on the system one can to provide a robust controller for the system. References and related issues of such problems for descriptor systems can be found in, e.g. [29], [10] and [15].

We consider in this section the $\mathcal{H}_\infty$ disturbance attenuation problem, i.e., the problem to find an admissible controller $u^* \in U_s$ for an attenuation level γ such that $\|G_{zw}\|_\infty < \gamma$. This problem is translated into a zero-sum linear quadratic differential game as finding an optimal control $u^*(t) = F_1^* x(t) \in U_s$ that satisfies

$$\inf_{u \in U_s} \sup_{w \in L_2(0,\infty)} \int_0^\infty \left[x^T(t)\bar{Q}x(t) + u^T(t)\bar{R}_1 u(t) - \gamma w^T(t)\bar{R}_2 w(t) \right] dt \tag{23}$$

or, equivalently,

$$\inf_{\tilde{F}_1} \sup_{\tilde{F}_2} \int_0^\infty \left\{ x_1^T(t) \begin{bmatrix} I & \tilde{F}_1^T & \tilde{F}_2^T \end{bmatrix} \tilde{M}_\gamma \begin{bmatrix} I \\ \tilde{F}_1 \\ \tilde{F}_2 \end{bmatrix} x_1(t) \right\} dt \tag{24}$$

subject to dynamical equation (1) with $x(0) = 0$.

We will present the solution to this problem that follows from the solution of the corresponding soft-constrained game considered in the previous section under the assumption that $\tilde{R}_{22\gamma} > 0$. From Theorem 2 we know that for every $\gamma > \gamma_{CL}^\infty$ the soft-constrained differential game has a FBSP solution where the minimizing controller is denoted by $u_\gamma^*(t) = F_{1\gamma}^* x(t)$. Furthermore, because of the existence of a conjugate point to RDE (16) on the interval $[0, t_f]$ as $\gamma \to \gamma_{CL}^\infty$ the limit of $u_\gamma^*(.)$ may not be well-defined. Hence in this paper we will only consider the suboptimal solution. To that end, let $\varepsilon > 0$ be sufficiently small and define $\gamma_\varepsilon := \gamma_{CL}^\infty + \varepsilon$. Considering the fact above and applying Theorem 3 in [5] and also Theorem 2, Theorem 3 to the problem (1,23) above results in the following theorem (see also Theorem 4.11 in [1] for the nonsingular game).

Theorem 4 *Consider the robust suboptimal control design (disturbance attenuation) problem* (1,23) *with feedback information structure and* $x_0 = 0$. *Let* (A, B) *be stabilizable,* $\left(A_1, \tilde{Q}^{\frac{1}{2}}\right)$ *be detectable,* $\tilde{Q} \geq 0$, $\tilde{R}_{11} > 0, \tilde{R}_{22\gamma} > 0$ *and assume that* (6) *holds. Then,*

$$\gamma^* = \inf_{\tilde{F}_1} \sup_{\tilde{F}_2} \int_0^\infty \left\{ x_1^T(t) \begin{bmatrix} I & \tilde{F}_1^T & \tilde{F}_2^T \end{bmatrix} \tilde{M}_\gamma \begin{bmatrix} I \\ \tilde{F}_1 \\ \tilde{F}_2 \end{bmatrix} x_1(t) \right\} dt = \gamma_{CL}^\infty \tag{25}$$

that is the minimax attenuation level is equal to γ_{CL}^∞. *Moreover, given any* $\varepsilon > 0$, *we have the bound*

$$\sup_{F_2} J_\gamma \left(F_{1\gamma_\varepsilon}^*, F_2 \right) \leq \gamma_\varepsilon := \gamma_{CL}^\infty + \varepsilon. \tag{26}$$

Here the robust suboptimal controller that achieves this bound is $u_{\gamma_\varepsilon}^{\infty *}(t) = F_{1\gamma_\varepsilon}^* x(t)$ where

$$F^*_{1\gamma_\varepsilon} = \tilde{F}^*_{1\gamma_\varepsilon} O^+ + Z_1\left(I - OO^+\right), \tag{27}$$

$Z_1 \in \mathbb{R}^{m_1\times(n+r)}$, $O = X\begin{bmatrix} I \\ -B_{12}\tilde{F}^*_{1\gamma_\varepsilon} - B_{22}\tilde{F}^*_{2\gamma_\varepsilon}\end{bmatrix}$, and $\left(\tilde{F}^*_{1\gamma_\varepsilon}, \tilde{F}^*_{2\gamma_\varepsilon}\right)$ are given by

$$\begin{bmatrix} \tilde{F}^*_{1\gamma_\varepsilon} \\ \tilde{F}^*_{2\gamma_\varepsilon} \end{bmatrix} = -\tilde{G}^{-1}_{\gamma_\varepsilon}\begin{bmatrix} B^T_{11}K^+_{\gamma_\varepsilon} + \tilde{V}^T \\ -B^T_{21}K^+_{\gamma_\varepsilon} - \tilde{W}^T \end{bmatrix} \tag{28}$$

where $K^+_{\gamma_\varepsilon}$ is the unique nonnegative definite solution of the ARE (14). Furthermore, for any $\varepsilon > 0$, $u^{\infty *}_{\gamma_\varepsilon}$ leads to a bounded input bounded state stable system.

If the detectability assumption is replaced by $\left(A_1, \tilde{Q}^{\frac{1}{2}}\right)$ has no unobservable modes on the imaginary axis then the same results as above holds, with only $K^+_{\gamma_\varepsilon}$ now taken as defined in Theorem 3, and the robust controller $u^{\infty *}_{\gamma_\varepsilon}$ restricted to the class of stabilizing control laws.

Proof Consider K_{γ_ε} as the nonnegative definite solution of Riccati differential equation (16) associated with γ_ε. Then, see e.g. [[4], Proposition 5.15], for $t_f \to \infty$, $\lim_{t_f\to\infty} K_{\gamma_\varepsilon} = K^+_{\gamma_\varepsilon}$ where $K^+_{\gamma_\varepsilon}$ is the nonnegative definite solution of ARE (14). Therefore, from Theorem 2 item 3, the differential game (1,7) satisfies

$$\inf_{F_1}\sup_{F_2} J_\gamma\left(F_1, F_2\right) = \sup_{F_2}\inf_{F_1} J_\gamma\left(F_1, F_2\right) = J_\gamma\left(F_1^*, F_2^*\right). \tag{29}$$

Since the game uses a feedback information framework, it is easily verified that $J_\gamma\left(u^*, w^*\right) = x_0^T M x_0$ for some matrix M. So, in particular we have $J_\gamma\left(u^*, w^*\right) = 0$ if $x_0 = 0$. Since $J_\gamma\left(u^*, w\right) \le J_\gamma\left(u^*, w^*\right)$ this implies

$$\int_0^{t_f} x^T_{u^*}(t)\bar{Q}x_{u^*}(t) + u^{*^T}(t)\bar{R}_1 u^*(t)\,dt \le \gamma\int_0^{t_f} w^T(t)\bar{R}_2 w(t)\,dt.$$

That is,

$$\left\|G_{u^*}(w)\right\|^2 \le \gamma\left\|w\right\|^2. \tag{30}$$

Next, According to (4) we can rewrite $\sup_{F_2} J_{\gamma_\varepsilon}\left(F^*_{1\gamma_\varepsilon}, F_2\right)$ as

$$\sup_{F_2} J_{\gamma_\varepsilon}\left(F^*_{1\gamma_\varepsilon}, F_2\right) = \sup_w \frac{\left\|G_{u^*_{\gamma_\varepsilon}}(w)\right\|}{\left\|w\right\|}.$$

Then, according to the inequality (30) for γ_ε, we have

$$\sup_{F_2} J_{\gamma_\varepsilon}\left(F^*_{1\gamma_\varepsilon}, F_2\right) = \sup_w \frac{\left\| G_{u^*_{\gamma_\varepsilon}}(w) \right\|}{\|w\|} \leq \sup_w \frac{\gamma_\varepsilon \|w\|}{\|w\|} = \gamma_\varepsilon,$$

from which it follows equation (26) holds. Next, because for $\gamma > \gamma_\varepsilon$ the game admits FBSP solutions then the robust suboptimal controller follows from Theorem 2 part 2. Further, since $K^+_{\gamma_\varepsilon}$ exists, using the results in part 5 of Theorem 2, the suboptimal robust controller then follows from (17) and (18). Finally, applying Theorem 3 to the problem (1,23) yields the last statement of Theorem 4. □

5. Concluding Remarks

This paper studies the robust optimal control problem for descriptor systems. Some theorems dealing with the problem have been constructed both on a finite and infinite planning horizon. The paper shows how the soft-constrained differential game formulation can be used to solve the disturbance attenuation problem and the key role played by the critical values γ^∞_{CL}.

The problem addressed in this paper is primarily restricted to the feedback information structure. An extension to other information structures, like delayed and sampled-data systems, and the case that the system is not perfect-state measurable seem to be interesting topics too, but this is an open problem left for future research.

Appendix. Transformation of the differential game

First, we state the following two lemmas that will be used in transforming the differential game. The first lemma can be found in [6].

Lemma 1 *Assume* $(E, A+BF)$ *is regular and has index one. Then for all* $F \in \mathcal{F}, G := I + \begin{bmatrix} B_{12} & B_{22} \end{bmatrix} FX_2$ *is invertible.*

Next, we recall from [3] the following lemma.

Lemma 2 *Assume* $C \in \mathbb{R}^{n\times m}$ *and* $D \in \mathbb{R}^{m\times n}$. *Then the following holds:*

1. $I_n + CD$ *is invertible if and only if* $I_m + DC$ *is invertible.*
2. *If* $I_n + CD$ *is invertible then* $C(I_m + DC)^{-1} = (I_n + CD)^{-1} C$.

To transform the differential game, consider the descriptor differential game described by the dynamical system

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + B_1u(t) + B_2w(t),\ x(0) = x_0 \\ y(t) &= C_1x(t) + D_{12}w(t) \\ z(t) &= C_2x(t) + D_{21}u(t). \end{aligned} \tag{31}$$

With the cost function preferred to be minimized by the first player

$$J_\gamma(u,w) = \int_0^\infty \left[x^T(t)\bar{Q}x(t) + u^T(t)\bar{R}_1u(t) - \gamma w^T(t)\bar{R}_2w(t)\right]dt \tag{32}$$

and for the second player $-J_\gamma(u,w)$, where $E, A \in \mathbb{R}^{(n+r)\times(n+r)}$, $rank(E) = n$, $B_i \in \mathbb{R}^{(n+r)\times m_i}$, $C_1 \in \mathbb{R}^{p\times(n+r)}$, $C_2 \in \mathbb{R}^{q\times(n+r)}$, $D_{12} \in \mathbb{R}^{p\times m_2}$, $D_{21} \in \mathbb{R}^{q\times m_1}$, $u(t) = F_1x(t) \in U_s$, $w(t) = F_2x(t) \in L_2(0,\infty)$, $y(t) \in \mathbb{R}^p$, and $z(t) \in \mathbb{R}^q$. By Weierstrass' canonical form (see e.g. [7]) there exists two nonsingular matrices X and Y such that

$$Y^TEX = \begin{bmatrix} I_n & 0 \\ 0 & N \end{bmatrix} \ and \ Y^TAX = \begin{bmatrix} J & 0 \\ 0 & I_r \end{bmatrix}.$$

Under the index one assumption, with $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} := X^{-1}x(t)$ where $x_1(t) \in \mathbb{R}^n$ and $x_2(t) \in \mathbb{R}^r$ the game (31,32) has an FBSP equilibrium actions (u^*, w^*) if and only if (u^*, w^*) are FBSP equilibrium actions for the game

$$\begin{aligned} \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} A_1 & 0 \\ 0 & I_r \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + Y^TB_1u(t) + Y^TB_2w(t), \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = X^{-1}x_0 \\ y(t) &= \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + D_{12}w(t) \\ z(t) &= \begin{bmatrix} \bar{C}_{21} & \bar{C}_{22} \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + D_{21}u(t), \end{aligned} \tag{33}$$

where the first player has the quadratic cost functional

$$J_\gamma(u,w) = \int_0^\infty \left[\begin{bmatrix} x_1^T(t) & x_2^T(t) \end{bmatrix} X^T\bar{Q}X\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + u^T(t)\bar{R}_1u(t) - \gamma w^T(t)\bar{R}_2w(t)\right]dt. \tag{34}$$

From (33) we use Lemma 1 for the infinite planning horizon to show that

$$\begin{aligned} x_2(t) &= -\begin{bmatrix} 0 & I_r \end{bmatrix}Y^T\left(B_1u(t) + B_2w(t)\right) \\ &= -\begin{bmatrix} 0 & I_r \end{bmatrix}Y^T\begin{bmatrix} B_1 & B_2 \end{bmatrix}\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}\begin{bmatrix} X_1 & X_2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \end{aligned}$$

From this we get, after some simple rewriting that

$$\begin{aligned} x_2(t) &= -\left(I+\begin{bmatrix} B_{12} & B_{22}\end{bmatrix} F X_2\right)^{-1}\begin{bmatrix} B_{12} & B_{22}\end{bmatrix} F X_1 x_1(t) \\ &=: H x_1(t). \end{aligned} \tag{35}$$

Substitution of (35) into the differential game (33,34) shows that (u^*, w^*) are FBSP equilibrium actions for the game (31,32) if and only if (F_1^*, F_2^*) are FBSP equilibrium actions for the game

$$\begin{aligned} \dot{x}_1(t) &= \left(A_1 + \begin{bmatrix} B_{11} & B_{12}\end{bmatrix}\begin{bmatrix} F_1 \\ F_2\end{bmatrix} X \begin{bmatrix} I \\ H\end{bmatrix}\right) x_1(t),\ x_1(0) = \begin{bmatrix} I_n & 0\end{bmatrix} X^{-1} x_0 \\ y(t) &= \bar{C}_{11} x_1(t) + \bar{C}_{12} x_2(t) + D_{12} w(t) \\ z(t) &= \bar{C}_{21} x_1(t) + \bar{C}_{22} x_2(t) + D_{21} u(t), \end{aligned} \tag{36}$$

with cost functional for the first player given by

$$\begin{aligned} J_\gamma(F_1,F_2) = \int_0^\infty \Bigg\{ & \begin{bmatrix} x_1^T(t) & x_1^T(t) H^T\end{bmatrix} X^T \bar{Q} X \begin{bmatrix} x_1(t) \\ x_1(t) H\end{bmatrix} \\ & + \begin{bmatrix} x_1^T(t) & x_1^T(t) H^T\end{bmatrix} X^T F_1^T \bar{R}_1 F_1(t) X \begin{bmatrix} x_1(t) \\ x_1(t) H\end{bmatrix} \\ & -\gamma \begin{bmatrix} x_1^T(t) & x_1^T(t) H^T\end{bmatrix} X^T F_2^T \bar{R}_2 F_2 X \begin{bmatrix} x_1(t) \\ x_1(t) H\end{bmatrix}\Bigg\} dt \end{aligned}$$

or, equivalently,

$$\begin{aligned} J_\gamma(F_1,F_2) = \int_0^\infty \Bigg\{ & x_1^T(t)\begin{bmatrix} I & H^T\end{bmatrix} X^T \begin{bmatrix} I & F_1^T & F_2^T\end{bmatrix}\begin{bmatrix} \bar{Q} & 0 & 0 \\ 0 & \bar{R}_1 & 0 \\ 0 & 0 & -\gamma \bar{R}_2\end{bmatrix} \\ & \times \begin{bmatrix} I \\ F_1 \\ F_2\end{bmatrix} X \begin{bmatrix} I \\ H\end{bmatrix} x_1(t)\Bigg\} dt. \end{aligned} \tag{37}$$

Now introducing

$$\tilde{F}_i := F_i X \begin{bmatrix} I \\ H\end{bmatrix}, \tag{38}$$

we can rewrite the game (36,37) in the form

$$\dot{x}_1(t) = \left(A_1 + \begin{bmatrix} B_{11} & B_{21}\end{bmatrix}\begin{bmatrix} \tilde{F}_1 \\ \tilde{F}_2\end{bmatrix}\right) x_1(t),\quad x_1(0) = \begin{bmatrix} I & 0\end{bmatrix} X^{-1} x_0 \tag{39}$$

and

$$J_\gamma\left(\tilde{F}_1,\tilde{F}_2\right)=\int_0^\infty\left\{x_1^T(t)\left[\left[I \quad H^T\right]X^T \quad \tilde{F}_1^T \quad \tilde{F}_2^T\right]\begin{bmatrix}\bar{Q} & 0 & 0\\ 0 & \bar{R}_1 & 0\\ 0 & 0 & -\gamma\bar{R}_2\end{bmatrix}\right.$$

$$\left.\times\begin{bmatrix}X\begin{bmatrix}I\\ H\end{bmatrix}\\ \tilde{F}_1\\ \tilde{F}_2\\ \end{bmatrix}x_1(t)\right\}dt \tag{40}$$

Next, notice that by Lemma 2,

$$\begin{aligned} H &= -\left(I+\left[B_{12} \quad B_{22}\right]FX_2\right)^{-1}\left[B_{12} \quad B_{22}\right]FX_1\\ &= -\left[B_{12} \quad B_{22}\right]F\left(I+X_2\left[B_{12} \quad B_{22}\right]F\right)^{-1}X_1\\ &= -\left(B_{12}\tilde{F}_1+B_{22}\tilde{F}_2\right). \end{aligned} \tag{41}$$

Using (41) shows that $\left(F_1^*,F_2^*\right)$ are FBSP equilibrium actions for the game (36,37) if and only if $\left(\tilde{F}_1^*,\tilde{F}_2^*\right)$ are FBSP equilibrium actions for the game (39) with cost function

$$J_\gamma\left(\tilde{F}_1,\tilde{F}_2\right)=\int_0^\infty\left\{x_1^T(t)\left[I \quad \tilde{F}_1^T \quad \tilde{F}_2^T\right]\tilde{M}_\gamma\begin{bmatrix}I\\ \tilde{F}_1\\ \tilde{F}_2\end{bmatrix}x_1(t)\right\}dt, \tag{42}$$

where

$$\tilde{M}_\gamma=\begin{bmatrix}\tilde{Q} & \tilde{V} & \tilde{W}\\ \tilde{V}^T & \tilde{R}_{11} & \tilde{N}\\ \tilde{W}^T & \tilde{N}^T & \tilde{R}_{22\gamma}\end{bmatrix}$$

and

$\tilde{Q}: X_1^T\bar{Q}X_1$, $\tilde{V}:=-X_1^T\bar{Q}X_2B_{12}$, $\tilde{W}:=-X_1^T\bar{Q}X_2B_{22}$, $\tilde{N}:=B_{12}^TX_2^T\bar{Q}X_2B_{22}$,
$\tilde{R}_{11}:=B_{12}^TX_2^T\bar{Q}X_2B_{12}+\bar{R}_1$, $\tilde{R}_{22\gamma}:=B_{22}^TX_2^T\bar{Q}X_2B_{22}-\gamma\bar{R}_2$.

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