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Robust Optimal Control Design Using a Differential Game Approach for Open-Loop Linear Quadratic Descriptor Systems

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Abstract This paper studies the robust optimal control problem for descriptor systems. We applied differential game theory to solve the disturbance attenuation problem. The robust control problem was converted into a reduced ordinary zero-sum game. Within a linear quadratic setting, we solved the problem for finite and infinite planning horizons.

Keywords Robust optimal control · Zero-sum linear quadratic differential game · Descriptor systems · Open-loop information structure

Mathematics Subject Classification 49N70 · 93B35

1 Introduction

Robust control concerns the design of controllers that perform acceptably for a family of systems under various types of inputs and disturbances, and not just for a single system and known inputs [1]. One technique for designing a robust controller uses a differential game approach [1–3]. A differential game is a mathematical model that represents a conflict between different agents that control a dynamical system. Every agent tries to minimize their individual objective function by controlling the

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system [4]. In the robust control framework, the control designer is the minimizing agent. They fight against disturbances and uncertainties, which are represented by the maximizing agent. This approach was used to find robust controllers for regular systems [1]. In this work, we applied this method to design robust optimal controllers for descriptor systems. We considered an open-loop information structure. The zero-sum linear quadratic descriptor differential game for index-one systems and this information structure was solved in [5], by transforming it into a regular differential game. We used the same procedure to solve the robust control design problem for descriptor systems. Merging results from [1] and [5], we first solved the robust optimal control problem for index-one descriptor systems. Next, we followed [6] and used these results to solve the problem for a broad class of higher index systems. [7] and [8] studied $\mathcal{H}_\infty$-optimal control for singularly perturbed systems. Part of their analysis studied the index-one case studied here.[1] They performed a number of state transformations to solve the problem. We avoided their detailed analysis by directly applying current, more developed research. This allows for a simplified presentation of the final results and, when there is an infinite planning horizon, further results for the case where the uncontrolled system is stable. Moreover, using this setting, we can directly analyze the higher-order index case.

In industrial applications that have online access to the system state, it is most appropriate to design robust controllers based on a feedback information structure. However, there are situations where a design based on an open-loop information structure is also appropriate, for instance, if the disturbances and uncertainties in the system are not too big and can be predicted before running the system. The advantage of such a control design is that it is less expensive, because the control can be implemented immediately for the whole planning horizon. Clearly, this type of strategy is often just an approximation of real-life strategic behavior. However, if punishment strategies may be enforced when policy makers do not stick to their commitments, we can realistically assume that players will stick to an agreement. For instance, in an economic policy setting, fiscal policies are typically conceived in a multi-annual budget framework, and institutions cannot adapt these budgets. Another example is calculating robust open-loop stable orbits for power generating kites, assuming that wind disturbances are random with a small bounded autocorrelation function [9].

The paper is organized as follows. In Sect. 2, we describe a reformulation of the robust optimal control problem as a differential game and translate the descriptor systems problem into a regular differential game. Section 3 contains a description of the corresponding zero-sum, index-one, descriptor differential game. We investigate finite- and infinite-horizon problems. We present our main results in Sect. 4 and discuss disturbance attenuation levels between $\gamma = 1$ and $\gamma \neq 1$. An example is given in Sect. 5, and Sect. 6 deals with the higher-order index case. Section 7 contains our conclusions.

[1] We would like to thank the referee pointing this out to us.

2 Problem Statement

Consider the dynamical system

$$\begin{aligned} E\dot{x}(t) &= A_E x(t) + B_u u(t) + B_w w(t), \; x(0) = x_0, \\ y(t) &= C_1 x(t) + D_{12} w(t), \\ z(t) &= C_2 x(t) + D_{21} u(t), \end{aligned} \tag{1}$$

where $E, A_E \in \mathbb{R}^{(n+r)\times(n+r)}$, rank $(E) = n$, $B_u \in \mathbb{R}^{(n+r)\times m_1}$, $B_w \in \mathbb{R}^{(n+r)\times m_2}$, $C_1 \in \mathbb{R}^{p\times(n+r)}$, $C_2 \in \mathbb{R}^{q\times(n+r)}$, $D_{12} \in \mathbb{R}^{p\times m_2}$, and $D_{21} \in \mathbb{R}^{q\times m_1}$. Here, $u \in U_s$, $w \in L_2^q\left(0, t_f\right)$, y, and z are the control variables, disturbance, measured output, and controlled output, respectively. U_s is the set of locally square integrable control functions yielding a stable closed-loop system. $L_2^q\left(0, t_f\right)$ is the set of all measurable Lebesgue square integrable functions on $\left(0, t_f\right)$. The control $u(t)$ is a linear mapping from a subset of the measured outputs $y(s)$, $s \le t$. The consistent initial state, x_0, is known (and is zero in the disturbance attenuation problem). (1) is called regular if and only if $\det(\lambda E - A_E) \neq 0$. The degree of nilpotency of N is denoted by r, and is an integer such that $N^r = 0$ and $N^{r-1} \neq 0$. The index of (1) is the degree r of nilpotency of N. If E is nonsingular, its index is zero. The above discussion motivates the assumptions that:

1. the matrix E is singular; and 2. $\det(\lambda E - A_E) \neq 0$.

From [10], we recall the Weierstrass canonical form of (1).

Theorem 2.1 *Assume that* (1) *is regular. Then, there exist nonsingular matrices X and Y such that*

$$Y^T E X = \begin{bmatrix} I_n & 0 \\ 0 & N \end{bmatrix} \text{ and } Y^T A_E X = \begin{bmatrix} J & 0 \\ 0 & I_r \end{bmatrix}. \tag{2}$$

Here, J is a matrix in Jordan form (that is, it has finite eigenvalues as elements), $I_k \in \mathbb{R}^{k\times k}$ is the identity matrix, and $N \in \mathbb{R}^{r\times r}$ is a nilpotent matrix that is also in Jordan form. J and N are unique up to a permutation of Jordan blocks.

Following [1], we next reformulate the robust optimal control problem as a differential game. Assume that the transfer function for (1) is $G_{zw} := \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$. Let $\hat{K}$ denote the transfer function of the controller. Each controller $\hat{K}$ induces a linear map $z = G_{zw} w$, where the transfer function G_{zw} is given by the linear fractional transformation (LFT), $G_{zw} = G_{11} + G_{12}\left(I - \hat{K}\cdot G_{22}\right)^{-1}\left(\hat{K}\cdot G_{21}\right)$. The design objective is to find a control that will keep the output (z) small, regardless of unpredictable disturbances (w). In mathematical terms: Given a positive number γ, find, if it exists, a controller $\hat{K}$ such that $\frac{\|z\|}{\|w\|} \le \gamma$. Or, in terms of the above LFT, find a controller such that the norm of the linear operator G_{zw} is smaller than γ. Let $G_u(w)$ be a bounded causal linear operator from w to z, i.e., $z = G_u(w)$. If there is a controller that makes the linear system stable, the induced linear operator norm of $G_{zw} = \sup_{w\in L_2^q(0,t_f)} \frac{\|G_u(w)\|}{\|w\|}$ exists and is equal to the $\mathcal{H}_\infty$ norm of G_{zw} (see, e.g., Proposition 1.1 in [1]). Then, the design problem can be reformulated into the following optimization problem. Find:

$$\inf_{u\in U_s} \|G_{zw}\|_\infty = \inf_{u\in U_s} \sup_{w\in L_2^q(0,t_f)} \frac{\|G_u(w)\|}{\|w\|}. \tag{3}$$

Denote this infimum by γ^*. Unfortunately, this infimum cannot be realized by choosing a specific stabilizing controller. The right-hand side of Eq. (3) defines an upper value for the game defined by (1) with objective function $\frac{\|G_u(w)\|}{\|w\|}$. Assume (see [1]) that there exists a control policy $u^* \in U_s$, and a corresponding γ^*, which satisfies (3). Then, (3) can be equivalently expressed as

1. there exist $u^* \in U_s$ such that $\|G_{u^*}(w)\|^2 \le \tilde{\gamma}^* \|w\|^2$, for all $w \in L_2^q\left(0, t_f\right)$, where $\tilde{\gamma} = \sqrt{\gamma}$; and
2. there exist no other $u \in U_s$(say, $\hat{u}$), and a corresponding $\hat{\gamma} < \gamma^*$, such that $\|G_{\hat{u}}(w)\|^2 \le \hat{\gamma}^* \|w\|^2$, for all $w \in L_2^q\left(0, t_f\right)$.

Now, consider a parameterized family of cost functions (in $\gamma \ge 0$),

$$J_\gamma(u, w) := \|G_u(w)\|^2 - \gamma \|w\|^2. \tag{4}$$

Then, 1 and 2 can be restated as finding the smallest $\gamma \ge 0$ under which the upper value of the game defined by (1) with objective function (4) is bounded above by zero, and a controller that achieves this upper value. Or, finding the minimal γ for which $\inf_{u\in U_s} \sup_{w\in L_2^q(0,t_f)} J_\gamma(u, w)$ exists.

Let $\|G_u(w)\|^2 := \int_0^{t_f} z^T(t) z(t)\,\mathrm{d}t + z^T\left(t_f\right) Q_{t_f} z\left(t_f\right)$, where $Q_{t_f} \ge 0$, and $\|w\|^2 := \int_0^{t_f} (Ww)^T(t)\, Ww(t)\,\mathrm{d}t$. Then, by combining Eq. (1) with the assumptions

$$C_2^T D_{21} = 0,\ C_2^T Q_{t_f} D_{21} = 0,\ \text{and}\ D_{21}^T Q_{t_f} D_{21} = 0, \tag{5}$$

the cost function (4) can be represented in a quadratic form,

$$\begin{aligned} J_\gamma(u, w) = &\int_0^{t_f} \left[x^T(t)\bar{Q}x(t) + u^T(t)\bar{R}_1 u(t) - \gamma w^T(t)\bar{R}_2 w(t)\right]\mathrm{d}t \\ &+ x^T\left(t_f\right)\bar{Q}_{t_f} x\left(t_f\right). \end{aligned} \tag{6}$$

Here, $\bar{Q}_{t_f} := C_2^T Q_{t_f} C_2$, $\bar{Q} := C_2^T C_2$, $\bar{R}_1 := D_{21}^T D_{21}$ and $\bar{R}_2 := W^T W$. This converts the robust control design problem into a linear quadratic zero-sum game. The game defined by Eqs. (1) and (6), with the cost function $J_\gamma(u, w)$ as the control designer, and $-J_\gamma(u, w)$ as “nature” (disturbances and uncertainties), is called the (zero-sum) soft-constrained differential game. This terminology is used to capture the feature of this game that there is no hard bound with respect to w [1]. So, for a fixed γ, the robust control design problem reduces to finding the optimal controller $u^* \in U_s$ that satisfies

$$\inf_{u \in U_s} \sup_{w \in L_2^q(0,t_f)} \int_0^{t_f} \left[x^T(t)\bar{Q}x(t) + u^T(t)\bar{R}_1 u(t) - \gamma w^T(t)\bar{R}_2 w(t) \right] \mathrm{d}t + x^T(t_f)\bar{Q}_{t_f}x(t_f),$$

subject to (1).

Next, assume that (1) has index one.[2] Consider (3), the Weierstrass canonical form of (1), and the corresponding state transformation $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} := X^{-1}x(t)$. Then, (u^*, w^*) is an open-loop saddle point (OLSP) solution for the index-one differential game in (1) and (6), if and only if $(u^* w^*)$ is an OLSP solution for the zero-sum differential game in (7) and (8) (see [5] (or [12,13]) for details). Details of the matrices below are presented in "Appendix."

The reduced dynamical system for the game is described by

$$\begin{aligned} \dot{x}_1(t) &= Ax_1(t) + B_1u(t) + B_2w(t), x_1(0) = \left[I_n \; 0 \right] X^{-1}x_0 =: x_{1_0}, \\ y(t) &= \bar{C}_{11}x_1(t) + \bar{C}_{12}x_2(t) + D_{12}w(t), \\ z(t) &= \bar{C}_{21}x_1(t) + \bar{C}_{22}x_2(t) + D_{21}u(t). \end{aligned} \tag{7}$$

The control designer wishes to minimize

$$J_\gamma(u, w) = \int_0^{t_f} \left\{ v^T(t) M_\gamma v(t) \right\} \mathrm{d}t + x_1^T(t_f) Q_{t_f} x_1(t_f), \tag{8}$$

with respect to u, where $v^T(t) := \left[x_1^T(t) \; u^T(t) \; w^T(t) \right]$ and $M_\gamma := \begin{bmatrix} Q & V & W \\ V^T & R_{\overline{11}} & N \\ W^T & N^T & R_{\overline{22}\gamma} \end{bmatrix}$.

And "nature" wishes to minimize $-J_\gamma(u, w)$ with respect to w.

3 The Open-Loop Zero-Sum Linear Quadratic Soft-Constrained Descriptor Differential Game

To design our robust optimal controllers, we recalled some results from [13] concerning open-loop zero-sum linear quadratic soft-constrained differential games for index-one descriptor systems. Corresponding "hard-constrained" results can be found in [14–17]. We suppose that the players, the control designer, and "nature" act noncooperatively. They only know the initial state and model structure.

[2] The system (1) has at most index one, if and only if $rank\left(\left[E \;\; A_E W \right]\right) = n + r$ (see [11]).

3.1 Finite Planning Horizon

Consider the game in (7) and (8) under the assumption that t_f is finite. The Riccati differential equations (RDEs) are important when finding the OLSP solution. They are

$$\dot{K}_2(t) = -A^T K_2(t) - K_2(t) A - (K_2(t) B_2 - W) R_{\overline{22}\gamma}^{-1} \left(B_2^T K_2(t) - W^T \right) + Q,$$
$$K_2(t_f) = -Q_{t_f}, \tag{9}$$
$$\dot{P}(t) = -\tilde{A}_\gamma^T P(t) - P(t) \tilde{A}_\gamma + P(t) B \hat{G}_\gamma^{-1} B^T P(t) - \hat{Q}_\gamma, \; P(t_f) = Q_{t_f}. \tag{10}$$

Here, $\tilde{A}_\gamma := A - B\bar{G}_\gamma^{-1}\bar{Z}$, $B := \begin{bmatrix} B_1 & B_2 \end{bmatrix}$, $\hat{G}_\gamma := \begin{bmatrix} R_{\overline{11}} & N \\ N^T & R_{\overline{22}\gamma} \end{bmatrix}$, $\bar{Z} := \begin{bmatrix} V^T \\ -W^T \end{bmatrix}$; $\tilde{Z} := \begin{bmatrix} V & W \end{bmatrix}$, and $\hat{Q}_\gamma := Q - \tilde{Z}\bar{G}_\gamma^{-1}\bar{Z}$. To ensure the RDE in (9) have a solution, we introduce the (nonempty) set

$$\Gamma_{OL} = \{\inf \Gamma_1, \inf \Gamma_2\}, \quad \gamma^* = \max \Gamma_{OL}, \tag{11}$$

where $\Gamma_1 = \left\{\tilde{\gamma} > 0 \,\middle|\, \forall \gamma > \tilde{\gamma}, \text{ the RDE (9) has no conjugate point on } [0, t_f] \right\}$, and $\Gamma_2 = \left\{\tilde{\gamma} > 0 \,\middle|\, \forall \gamma > \tilde{\gamma}, -\bar{R}_{\overline{22}\gamma} > 0 \right\}$.[3] Our first result is as follows (see [13] for a detailed proof).

Theorem 3.1 *Consider the differential game in* (7) *and* (8). *Assume that* (5) *holds;* $\bar{R}_i > 0$*; and* $R_{\overline{22}\gamma} < 0$. *Then, we have the following.*

1. *For* $\gamma > \gamma^*$, *the RDEs* (9) and (10) *do not have a conjugate point on* $[0, t_f]$.
2. *For* $\gamma > \gamma^*$, *the game* (7), (8) *admits a unique OLSP solution for every initial state that allows for feedback synthesis, which is given by*

$$\begin{bmatrix} u^*(t) \\ w^*(t) \end{bmatrix} = -\bar{I}\bar{G}_\gamma^{-1} \left(\bar{Z} + \tilde{B}^T \bar{P}(t) \right) x_1(t). \tag{12}$$

Here, $\bar{I} := \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$; $\bar{Z} := \begin{bmatrix} V^T \\ -W^T \end{bmatrix}$; $\tilde{B}^T := \operatorname{diag}\{B_1^T, B_2^T\}$; $\bar{P} := \begin{bmatrix} P \\ P \end{bmatrix}$; $P(t)$ *solves RDE* (10)*; and* $x_1(t)$ *satisfies the differential equation*

$$\dot{x}_1(t) = \left(A - B\bar{G}_\gamma^{-1} \left(\bar{Z} + \tilde{B}^T \bar{P}(t) \right) \right) x_1(t), \; x_1(0) = \begin{bmatrix} I_n & 0 \end{bmatrix} X^{-1} x_0. \tag{13}$$

3. *If* $R_{\overline{22}\gamma^*} < 0$ *for* $\gamma \le \gamma^*$, *the upper value of the game in* (7) *and* (8) *is unbounded for any* $x_0 \in \mathbb{R}^n$. *Furthermore, there is some* $x_0 \in \mathbb{R}^n$ *for which the upper value is also unbounded at* $\gamma = \gamma^*$.

Remark 3.1 If $R_{\overline{22}\gamma^*}$ is singular, $J_\gamma(u, w)$ is not strictly concave in w for every u. In this case, the solution is not unique. A more detailed analysis of this case remains a subject for further research.

[3] We write $X > 0 (X \ge 0)$ if X is positive (semi) definite.

3.2 Infinite Planning Horizon

In this section, we assume that the control designer wants to minimize

$$\begin{aligned} J_\gamma(u,w) &= \int_0^\infty \left\{ x^T(t)\bar{Q}x(t) + u^T(t)\bar{R}_1 u(t) - \gamma w^T(t)\bar{R}_2 w(t) \right\} \mathrm{d}t \\ &= \int_0^\infty \left\{ v^T(t) M_\gamma v(t) \right\} \mathrm{d}t, \end{aligned} \tag{14}$$

with respect to u. "Nature" wants to minimize $-J_\gamma(u,w)$ with respect to w. We assume that the matrix pairs (A_E, B_i), $i = u, w$ in (1) are finite dynamics stabilizable,[4] i.e., $rank\,([\lambda E - A_E, B_i]) = n + r, \forall \lambda \in \mathbb{C}_0^+$. It can be easily shown that this assumption is equivalent to the assumption that the matrix pairs (A, B_i), $i = 1, 2$ are stabilizable in (7). So, in principle, each player could stabilize the system in (1) on their own. Furthermore, we assume that the matrix pair $(A_E, \bar{Q})$ in (1) and (6) is finite dynamics detectable, i.e., $rank\left(\begin{bmatrix} \lambda E - A_E \\ \bar{Q} \end{bmatrix}\right) = n + r, \forall \lambda \in \mathbb{C}_0^+$. Or, equivalently, the matrix pair (A, Q) is detectable in (7) and (8). The admissible set of control functions is

$$U_s(x_{1_0}) := \left\{ u \in L_2(0,\infty) \left| \begin{array}{l} \lim\limits_{t_f \to \infty} J_\gamma(t_f, x_{1_0}, u) \in \mathbb{R} \cup \{-\infty, \infty\}, \\ \lim\limits_{t \to \infty} x_1(x_{1_0}, u, t) = 0 \end{array} \right. \right\}. \tag{15}$$

For notational simplicity, we omit the dependency of U_s on x_{1_0}.[5]

To solve the game, we consider the optimization problem

$$\sup_{w \in L_2^q(0,\infty)} J_\gamma(0, w). \tag{16}$$

Instead of RDEs (9) and (10), we consider the associated algebraic Riccati equations (AREs):

$$0 = A^T K_2(t) + K_2(t) A + (K_2(t) B_2 - W) R_{22\gamma}^{-1} \left(B_2^T K_2(t) - W^T \right) - Q, \tag{17}$$

$$0 = \tilde{A}_\gamma^T P + P\tilde{A}_\gamma + PB\hat{G}_\gamma^{-1} B^T P - \hat{Q}_\gamma. \tag{18}$$

Furthermore, we define $\Gamma_{OL}^\infty = \left\{ \inf \Gamma_1^\infty, \inf \Gamma_2^\infty \right\}$ and $\gamma_{OL}^\infty = \max \Gamma_{OL}^\infty$, where $\Gamma_1^\infty = \{\tilde{\gamma} > 0 \,|\, \forall \gamma > \tilde{\gamma}$, if ARE (17) has a nonnegative definite solution$\}$ and $\Gamma_2^\infty = \left\{ \tilde{\gamma} > 0 \,\middle|\, \forall \gamma > \tilde{\gamma}, \text{if } -\bar{R}_{22\gamma} > 0 \right\}$.

[4] $\mathbb{C}^- = \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}(\lambda) < 0\}$ $\mathbb{C}_0^+ = \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}(\lambda) \geq 0\}$.

[5] $\lim_{t_f \to \infty} J_i(t_f, x_{1_0}, u) = -\infty\ (\infty)$ if $\forall r \in \mathbb{R}, \exists T_f \in \mathbb{R}$, such that $t_f \geq T_f$ implies $J_i(t_f, x_{1_0}, u) \leq r\ (\geq r)$.

Because $Q \geq 0$ and (A, Q) is finite dynamics detectable, the optimization problem $\min_{u \in U_S} J_\gamma(u, w)$ has a unique solution for every fixed $w \in L_2^q(0, \infty)$. Theorem 3.2 implies that the optimization problem (16) has a solution (which can be proved in the same manner as [1, pp. 363–369]).

Theorem 3.2 *Assume that* $(A, B_i), i = 1, 2$ *is stabilizable and* (A, Q) *is detectable. Then, for* $\gamma > \gamma_{OL}^{\infty}$, $\sup_{w \in L_2^q(0,\infty)} J_\gamma(0, w) < \infty, \forall x_0 \in \mathbb{R}^n$. *Furthermore,* (17) *has a symmetric solution,* $K_{2\gamma}^{+}$, *such that*

$$\hat{A} = A - \gamma^{-2} B_2 R_{\overline{22}\gamma}^{-1} \left(-W^T + B_2^T K_{2\gamma}^{+} \right) \tag{19}$$

is stable.[6] *Then,*

1. $K_{2\gamma}^{+} \geq 0$, *and it is positive definite if the pair* $\left(A, Q^{\frac{1}{2}}\right)$ *is observable;*
2. $K_{2\gamma}^{+}$ *is the unique solution of* (17) *if it satisfies* (19)*;*
3. $K_{2\gamma}^{+}$ *is the minimal solution of* (17) *(i.e., if there is some other solution,* $\bar{K}_{2\gamma}$, *then* $\bar{K}_{2\gamma} - K_{2\gamma}^{+} \geq 0$)*;*
4. $\sup_{w \in L_2^q(0,\infty)} J_\gamma(0, w) = x_0^T K_{2\gamma}^{+} x_0$*;*
5. $K_{2\gamma}^{+} = \lim_{t_f \to \infty} K_{2\gamma}\left(t, t_f\right)$ *for* $Q_{t_f} = 0$, *where* $K_{2\gamma}$ *solves RDE* (9) *for* $\gamma > \gamma_{OL}^{\infty}$.

Moreover, if $R_{\overline{22}\gamma_{OL}^{\infty}} < 0$ *for* $\gamma < \gamma_{OL}^{\infty}$, *the value in* (16) *is infinite and* (17) *has no real solution.*

Combining Theorem 3.2 with [5, Theorem 4.2] results in Theorem 3.3.

Theorem 3.3 *Consider the differential game in* (7) and (14). *Assume that* (5) *holds;* $\bar{R}_i > 0$; $R_{\overline{22}\gamma} < 0$; (A, B_i) *are stabilizable; and* (A, Q) *is detectable. Let* $\gamma > \Gamma_1^{\infty}$. *Then, this game has a unique OLSP solution for every initial state if and only if the following conditions hold.*

1. *ARE* (17) *has a solution such that* $\hat{A}$ *in* (19) *is stable.*
2. *The coupled AREs*

$$0 = \tilde{A}_\gamma^T P_1 + P_1 \tilde{A}_\gamma - P_1 B \hat{G}_\gamma^{-1} B^T P_2 + \hat{Q}_\gamma, \tag{20}$$

$$0 = \tilde{A}_\gamma^T P_2 + P_2 \tilde{A}_\gamma - P_2 B \hat{G}_\gamma^{-1} B^T P_1 - \hat{Q}_\gamma, \tag{21}$$

have a set of solutions $\tilde{P} = \left[P_1^T \;\; P_2^T \right]^T$ *such that* $A - B\bar{G}_\gamma^{-1}\left(\bar{Z} + \tilde{B}^T \tilde{P}\right)$ *and* $\begin{bmatrix} \tilde{A}^T & 0 \\ 0 & \tilde{A}^T \end{bmatrix} - PB\hat{G}_\gamma^{-1} \begin{bmatrix} B^T & 0 \\ 0 & B^T \end{bmatrix}$ *are stable.*[7]

Moreover, if a unique OLSP solution exists, the equilibrium actions are

$$\begin{bmatrix} u^*(t) \\ w^*(t) \end{bmatrix} = -\bar{G}_\gamma^{-1}\left(\bar{Z} + \tilde{B}^T \tilde{P}\right) x_1(t).$$

[6] Matrix A is called stable if the real parts of all its eigenvalues are negative.

[7] Such a solution is called an LRS solution.

Here, x_1 solves the differential equation

$$\dot{x}_1(t) = \left(A - B\bar{G}_\gamma^{-1}\left(\bar{Z} + \tilde{B}^T\tilde{P}\right)\right)x_1(t), \quad x_1(0) = \left[\,I_n \; 0\,\right]X^{-1}x_0.$$

If $R_{\overline{22}\gamma_{OL}^\infty} < 0$ for $\gamma < \gamma_{OL}^\infty$, the upper value of the game in (7) and (14) *is unbounded for any $x_0 \in \mathbb{R}^{n+r}$.*

The proof of Theorem 3.4 requires the following lemma, which was proved in [13].

Lemma 3.1 *Equation* (18) *has a stabilizing solution if* (17) *has a stabilizing solution.*

If matrix A is stable,[8] the stabilizability and detectability assumption are trivially satisfied. It can be shown, along the lines of the proof of Proposition 7.20 in [18], that the coupled AREs (20) and (21) have a solution, if and only if (18) has a stabilizing solution. Lemma 3.1 implies that its solution is guaranteed if (17) has a stabilizing solution. This leads to the next specialization of Theorem 3.3. The proof can be found in corresponding part of Proposition 7.20 in [18].

Theorem 3.4 *Consider the differential game in* (7) *and* (14)*. Assume that* (5) *holds; A is stable; $\bar{R}_i > 0$; and $R_{\overline{22}\gamma} < 0$.*

1. *For $\gamma > \gamma_{OL}^\infty$, AREs* (17) *and* (18) *have a symmetric solutions P and K, such that for $\bar{P} = \left[\,P^T \; P^T\,\right]^T$, both $A - B\bar{G}_\gamma^{-1}\left(\bar{Z} + \tilde{B}^T\bar{P}\right)$ and $A + B_2R_{\overline{22}\gamma}^{-1}\left(-W^T + B_2^T K\right)$ are stable.*
2. *For $\gamma > \gamma_{OL}^\infty$, the game admits a unique OLSP solution that allows for a feedback synthesis, which is given by*

$$\begin{bmatrix} u^*(t) \\ w^*(t) \end{bmatrix} = -\bar{I}\bar{G}_\gamma^{-1}\left(\bar{Z} + \tilde{B}^T\bar{P}\right)x_1(t). \tag{22}$$

Here, x_1 satisfies the differential equation

$$\dot{x}_1(t) = \left(A - B\bar{G}_\gamma^{-1}\left(\bar{Z} + \tilde{B}^T\bar{P}\right)\right)x_1(t), \; x_1(0) = \left[\,I_n \; 0\,\right]X^{-1}x_0.$$

3. *For $\gamma > \gamma_{OL}^\infty$, the cost for the control designer is*

$$L_\gamma = \left(\left[\,I \; 0\,\right]X^{-1}x_0\right)^T P\left[\,I \; 0\,\right]X^{-1}x_0, \tag{23}$$

and the cost for "nature" is $-L_\gamma$.

[8] Note from (2) that the assumption that A is stable is equivalent to the assumption that all finite eigenvalues of A_E are stable.

4. *If* $R_{\overline{22}\gamma_{OL}^{\infty}} < 0$ *for* $\gamma < \gamma_{OL}^{\infty}$*, the upper value of the game in* (7) *and* (14) *is unbounded for any* $x_0 \in \mathbb{R}^n$*.*

4 Robust Optimal Control Design

Robust control aims to design controllers such that the performance or characteristics of the systems can be maintained for all allowable parameter uncertainties. References and related issues for descriptor systems can be found in, e.g., [19–22]. In this section, we consider the $\mathcal{H}_\infty$ disturbance attenuation problem. That is, for attenuation level γ, find an admissible controller $u^* \in U_s$ such that $\|G_{zw}\|_\infty < \gamma$. As shown in Sect. 2, this problem can be translated into a zero-sum linear quadratic differential game, where the dynamics of the game are given by (7), with a zero initial state. In this setup, γ also can be interpreted as a parameter that models the expectation of the control designer with respect to the amount of noise that will enter the system. Therefore, we also considered the game with a nonzero initial state.

4.1 Finite Planning Horizon

Here, we consider the problem of finding an optimal control $u^* \in U_s$ that solves

$$\inf_{u\in U_s} \sup_{w\in L_2^q(0,t_f)} \int_0^{t_f} \left\{v^T(t) M_\gamma v(t)\right\} \mathrm{d}t + x_1^T\left(t_f\right) Q_{t_f} x_1\left(t_f\right), \tag{24}$$

subject to dynamical system (7). To guarantee that RDE (9) does not have a conjugate point for a level of disturbance attenuation γ, we use (11). We assume that, if an open-loop robust optimal control exists, the control designer can implement it as a feedback controller. In that case, the dynamics of the game are described by (13). The following theorem is an immediate consequence of Theorem 3.1.

Theorem 4.1 *Consider the robust optimal control design problem (or the disturbance attenuation problem): find* (24) *subject to* (7)*. Assume that* (5) *holds;* $\bar{R}_i > 0$*; and* $R_{\overline{22}\gamma^*} < 0$*. Then, we have the following.*

1. *The optimal attenuation level is* γ^**.*
2. *For* $\gamma > \gamma^*$*, RDE* (9) *does not have a conjugate point on* $\left[0, t_f\right]$*.*
3. *For* $\gamma > \gamma^*$*, the problem in* (7) *and* (24) *has a unique robust optimal controller* (12)*.*
4. *If* $R_{\overline{22}\gamma^*} < 0$ *for* $\gamma < \gamma^*$*, the upper value of the game in* (7) *and* (24) *is unbounded for any* $x_0 \in \mathbb{R}^n$*. Furthermore, there is some* $x_0 \in \mathbb{R}^n$ *for which the upper value is also unbounded at* $\gamma = \gamma^*$*.*

Proof Assume that $\gamma > \gamma^*$. Theorem 3.1 implies that the differential game in (7) and (8) has a unique saddle point solution for every initial state. Therefore, the game satisfies

$$\inf_{u \in U_s} \sup_{w \in L_2^q(0,t_f)} J_\gamma(u, w) = \sup_{w \in L_2^q(0,t_f)} \inf_{u \in U_s} J_\gamma(u, w) = J_\gamma\left(u^*, w^*\right).$$

So, the solution of the problem in (7) and (24) coincides with the solution for (7) and (8), that is, $J(u^*, w^*)$. From this, the conclusion is obvious. □

The above theorems show (see also [18]) that, in this open-loop disturbance attenuation problem, the best open-loop worst-case robust controller is $u = 0$, whereas the worst-case signal is $w = 0$. So, if a stable system is in equilibrium (i.e., $x_0 = 0$), it is best not to react to potential unknown disturbances in this open-loop framework. However, in case the current state of the system is available for control purposes, we can use Item 3 of Theorem 4.1, to implement the controller as a state feedback controller. This is, of course, much more robust.

4.2 Infinite Planning Horizon

In this section, we derive $u^* \in U_s$ that solves

$$\inf_{u \in U_s} \sup_{w \in L_2^q(0,\infty)} \int_0^\infty \left\{v^T(t) M_\gamma v(t)\right\} \mathrm{d}t, \tag{25}$$

subject to dynamical equation (7), with $x(0) = 0$. The assumptions made throughout this subsection are: the matrix pairs (A, B_i), $i = 1, 2$ are stabilizable; (A, Q) are detectable; and the controls belong to the set of control functions (15). Theorem 3.4 leads to the next result.

Theorem 4.2 *Consider the robust optimal control design problem (disturbance attenuation problem): find* (25) *subject to* (7)*. Assume that* (5) *holds; A is stable; $\bar{R}_i > 0$; and $R_{\overline{22}\gamma_{OL}^\infty} < 0$. Then, we have the following.*

1. $\inf_{u \in U_s} \sup_{w \in L_2^q(0,\infty)} \int_0^\infty \left\{v^T(t) M_\gamma v(t)\right\} \mathrm{d}t = \gamma_{OL}^\infty$. *That is, the optimal attenuation level equals γ_{OL}^∞.*
2. *For $\gamma > \gamma_{OL}^\infty$, ARE* (18) *has a symmetric solution P such that $A - B\bar{G}_\gamma^{-1}\left(\bar{Z} + \tilde{B}^T \bar{P}\right)$ is stable, and ARE* (17) *has a symmetric solution K_2 such that $A - B_2 R_{22\gamma}^{-1}\left(-W^T + B_2^T K_2\right)$ is stable.*
3. *For $\gamma > \gamma_{OL}^\infty$, the problem in* (7) *and* (25) *has a unique robust optimal controller* (22).
4. *For $\gamma > \gamma_{OL}^\infty$, the optimum value for the problem in* (25) *and* (7) *is* (23).
5. *If $R_{\overline{22}\gamma_{OL}^\infty} < 0$ for $\gamma < \gamma_{OL}^\infty$, the upper value of the game in* (7 and 25) *is unbounded for any $x_0 \in \mathbb{R}^n$.*

Proof The proof is similar to the proof of Theorem 4.1, with $J_\gamma(u, w) = \int_0^\infty \left\{v^T(t) M_\gamma v(t)\right\} \mathrm{d}t$. □

5 Numerical Example

Consider the robust optimal control design problem: find

$$\inf_{u \in U_s} \sup_{w \in L_2(0,\infty)} \int_0^\infty \left\{ x^T(t)\,\bar{Q}x(t) + u^T(t)\,\bar{R}_1 u(t) - \gamma w^T(t)\,\bar{R}_2 w(t) \right\} \mathrm{d}t,$$

subject to the dynamical system (1), where $E = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$, $A_E = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$, $B_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $B_w = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\bar{Q} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$, $\bar{R}_1 = [1]$, $\bar{R}_2 = [3]$, $C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $D_{12} = [1]$, $D_{21} = [1]$, and the initial state $x_0 = \frac{1}{2}[1 \, -1]^T$. Using Weierstrass' canonical form [Eq. (2)] with $Y^T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, the above robust optimal control problem is equivalent to the problem: find (25) subject to the dynamical system (7), where $M_\gamma = \begin{bmatrix} 3 & -4 & -4 \\ -4 & 7 & 6 \\ -4 & 6 & 6-3\gamma \end{bmatrix}$, $X_1 = Y_2^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $X_2 = Y_1^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $N = [0]$, $J = [1]$, $\bar{C}_{11} = [1]$, $\bar{C}_{12} = [1]$, $\bar{C}_{21} = [1]$, and $\bar{C}_{22} = [1]$. Theorem 4.2 implies that this problem has a solution if the ARE

$$(-24 + 12\gamma)\,k^2 + (-94 + 48\gamma)\,k - 99 + 48\gamma = 0$$

has a stabilizing solution k. This is the case for $\gamma > \frac{73}{36}$. Furthermore, we can easily verify that $R_{\overline{22}\gamma} < 0$ for $\gamma > \frac{73}{36}$. So, using (22), the robust controller for this problem is

$$u^*(t) = \frac{-12\gamma + (-18 + 3\gamma)\,p}{6 - 21\gamma} e^{\frac{-14+33\gamma-(22+3\gamma)p}{-6+21\gamma}t}, \tag{26}$$

and the closed-loop system in the case of the worst disturbance occurs is

$$\dot{x}_1(t) = \frac{-14 + 33\gamma - (22 + 3\gamma)\,p}{-6 + 21\gamma} x_1(t), \quad x_1(0) = 1.$$

Here, p is a solution to the ARE

$$\frac{10 - 3\gamma}{6 - 21\gamma} p^2 + \left(2 + \frac{-28 + 66\gamma}{6 - 21\gamma}\right) p + \frac{-34 + 111\gamma}{6 - 21\gamma} = 0.$$

Figure 1 illustrates the optimal trajectory of state $x_1^*(t)$ when the control designer uses the robust controller in Eq. (26) for different values of γ, if the corresponding worst-case disturbance w occurs. This figure shows that a smaller γ leads to a quicker stabilization of the system. Figure 2 illustrates the corresponding robust optimal controller $(u^*(t))$ for different values of γ. This figure shows the resulting high gain property of the controllers when γ is small. Furthermore, Figs. 3 and 4 show

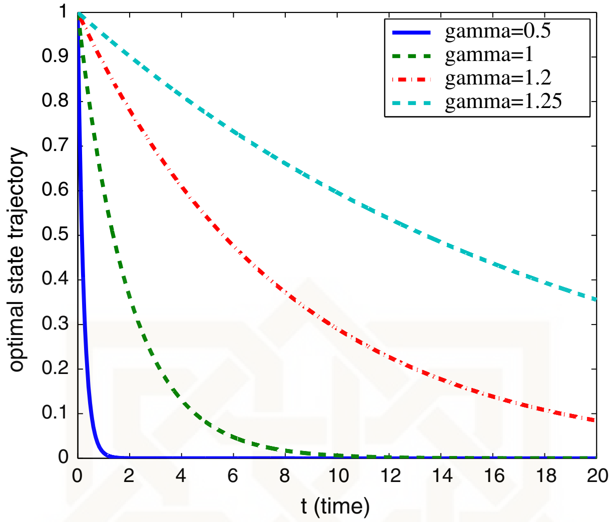

Fig. 1 Optimal trajectory of state $x_1^*(t)$

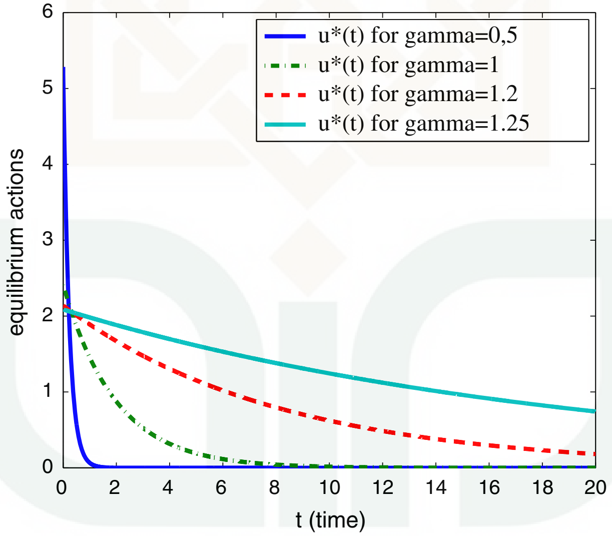

Fig. 2 Robust controller $(u^*(t))$ and disturbance $(w^*(t))$

the resulting closed-loop responses when the system is affected by a non-worst-case disturbance or a non-admissible disturbance. Figure 3 shows that the system stabilized more quickly, when compared with Fig. 1. Conversely, Fig. 4 shows that the controller did not stabilize the state trajectory subject to the non-admissible disturbance.

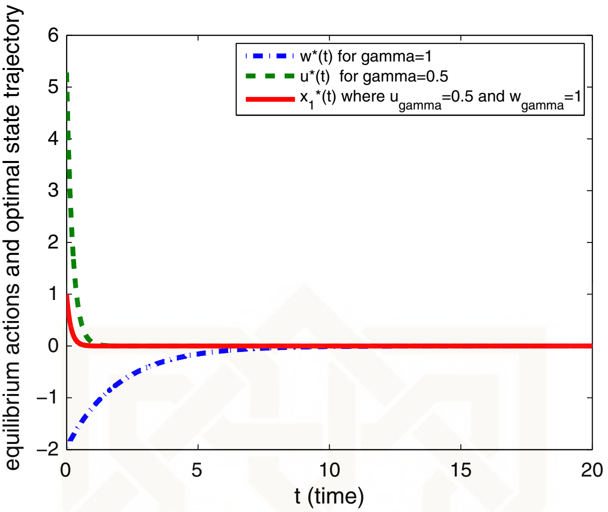

Fig. 3 Action $u^*_{\gamma=0.5}$ and $w^*_{\gamma=1}$ (t) for x^*_1 (t)

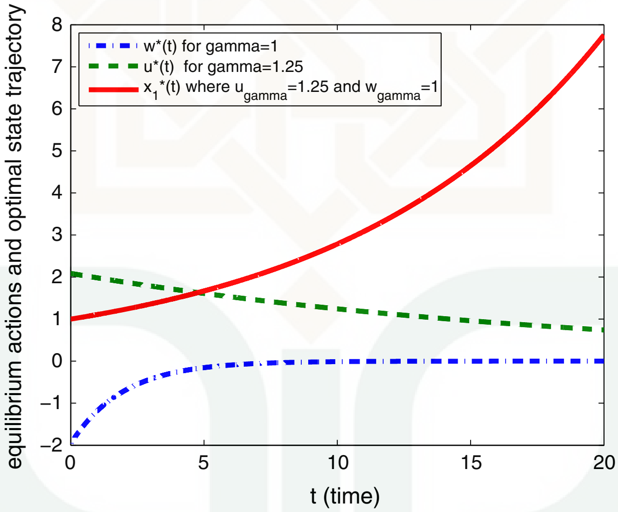

Fig. 4 Action $u^*_{\gamma=1.25}$ and $w^*_{\gamma=1}$ (t) for x^*_1 (t)

6 The Higher-Order Index Case

In this section, we briefly discuss the case when system (1) has an index k that is larger than one. We may proceed using index reduction algorithms, which have been proposed to reduce higher-order index systems to index-one systems (see, e.g., [23] and [24]). Assume that the corresponding disturbance attenuation problem can be similarly transformed into a disturbance attenuation problem for a system of index one. Then, the theory developed in the previous sections can be used to address robustness issues for such higher-order index systems. We, did not take this approach in this paper. Instead, we followed [6], which meant we could directly apply the theory developed in the previous sections.

If a system is of index k, the state trajectories typically include $(k-1)$th order derivatives of the applied input. For that reason, we consider the $(k-1)$th order derivative of the applied input as the control instrument, and also consider the possibility that the controlled output z depends on the derivatives of the applied input. We restricted our analysis to the case when only the dynamic part of the system is affected by noise. This is an appropriate assumption for most applications. When the non-dynamic part of the system is also corrupted by noise, we can proceed in a similar way by considering the $(k-1)$th order derivative of the noise as the control instrument of "nature."

Assuming that system (1) has index $k > 1$, let $u^{(i)} := \frac{d^i u(t)}{dt^i}, i = 0, \ldots, k-2$; $\tilde{u}(t) := \frac{d^{(k-1)}u(t)}{dt^{(k-1)}}$; and $z(t) = C_2 x(t) + \sum_{j=0}^{k-2} C_{2j} u^{(j)}(t) + D_{21}\tilde{u}(t)$. Here, $C := [C_2\ C_{20}\ \cdots C_{2k-2}]$, $C^T D_{21} = 0$, $C^T Q_{tf} D_{21} = 0$, and $D_{21}^T Q_{t_f} D_{21} = 0$.

Next, consider the discounted version of the previously introduced norms, i.e.,

$$\|G_u(w)\|^2 := \int_0^{t_f} e^{-\theta t} z^T(t)\, z(t)\,\mathrm{d}t + e^{-\theta t_f} z^T(t_f)\, Q_{t_f} z(t_f),$$

$$\|w\|^2 := \int_0^{t_f} e^{-\theta t} (Ww(t))^T\, Ww(t)\,\mathrm{d}t.$$

The discount factor, θ, is assumed to be nonnegative and, if the planning horizon is infinite, strictly positive.

Introducing $v^{e^T}(t) := [x^T(t)\ u^T(t)\ \cdots u^{(k-2)^T}]$, the parameterized family of cost functions $J_\gamma(u, w) := \|G_u(w)\|^2 - \gamma\|w\|^2$ can be rewritten as

$$J_\gamma(\tilde{u}, w) = \int_0^{t_f} e^{-\theta t}\left[v^{e^T}(t)\,\bar{Q}_z v^e(t) + \tilde{u}^T(t)\,\bar{R}_{z1}\tilde{u}(t) - \gamma w^T(t)\,\bar{R}_{z2} w(t)\right]\mathrm{d}t$$
$$+ e^{-\theta t} x^T(t_f)\,\bar{Q}_{zt_f} x(t_f).$$

Or, using the notation $x^{e^T}(t) := [v^{e^T}(t)\ \tilde{u}^T\ w^T]$ and $\bar{M}_\gamma = H_1^T H_1 - \gamma H_2^T H_2$, where $H_1 := [C\ D_{21}\ 0_{q\times m_2}]$ and $H_2 := [0_{q\times(n+r+km_1)}\ W]$,

$$J_\gamma(\tilde{u}, w) = \int_0^{t_f} e^{-\theta t}\left\{x^{e^T}(t)\bar{M}_\gamma x^e(t)\right\}\mathrm{d}t + e^{-\theta t_f} x^T(t_f)\bar{Q}_{zt_f} x(t_f). \quad (27)$$

As mentioned above, we assume that only the dynamic part of the system is affected by noise, i.e., $Y_2 B_2 = 0$. Then, if $\left[x_1^T(t)\ x_2^T(t)\right]^T := X^{-1}x$ where $x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^r$, the system dynamics in (1) can be rewritten for any consistent initial state as:

$$\dot{x}_1(t) = Jx_1(t) + Y_1B_uu(t) + Y_1B_ww(t),\ x_1(0) = [I\ 0]X^{-1}x_0, \tag{28}$$

$$x_2(t) = -\sum_{i=0}^{k-1} N^iY_2B_uu^{(i)}(t) = -\sum_{i=0}^{k-2} N^iY_2B_uu^{(i)}(t) - N^{k-1}Y_2B_u\tilde{u}(t). \tag{29}$$

Next, we introduce the discounted state, control, and disturbance vectors defined as $x_z^T := e^{-\frac{1}{2}\theta t}[x_1^T(t)u^T(t)\ \cdots u^{(k-2)^T}(t)], u_1(t) := e^{-\frac{1}{2}\theta t}\tilde{u}^T(t), w_1(t) := e^{-\frac{1}{2}\theta t}$ $w^T(t)$, respectively. Additionally, $z^T(t) := [x_z^T(t)\ u_1^T(t)w_1^T(t)]$. Then, if $m := km_1 + m_2$, $P_1 := [Y_2B_u\ NY_2B_u\ \cdots\ N^{k-2}Y_2B_u]$, $Z_1 := [I_n\ 0_{n\times m}]$, and $Z_2 := -[0_{r\times n}\ P_1\ N^{k-1}Y_2B_u\ 0_{r\times m_2}]$, we have

$$e^{-\frac{1}{2}\theta t}x(t) = X\begin{bmatrix} e^{-\frac{1}{2}\theta t}x_1(t) \\ e^{-\frac{1}{2}\theta t}x_2(t)\end{bmatrix} = X\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} z(t) =: L_1z(t).$$

Furthermore, we define $E_2 := [0_{m\times n}\ I_m]:\ e^{-\frac{1}{2}\theta t}x^e(t) = \left[L_1^T\ E_2^T\right]^T z(t) =: Lz(t)$. Next, let $A_1 := [B_1\ 0_{n\times(k-2)m_1}]$ and,

$$D_1 := \begin{bmatrix} -\frac{1}{2}\theta I & I & 0 & \cdots & 0 \\ 0 & & & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ & & & & I \\ 0 & & \cdots & 0 & -\frac{1}{2}\theta I \end{bmatrix} \in \mathbb{R}^{(k-1)m_1\times(k-1)m_1};\ \text{and}$$

$$B_{z_1} := \begin{bmatrix} 0 \\ \\ \vdots \\ 0 \\ I \end{bmatrix} \in \mathbb{R}^{(k-1)m_1\times m_1},$$

where $I \in \mathbb{R}^{m_1\times m_1}$. We can then easily verify that the zero-sum game in (1 and 27) has a set of OLN equilibrium actions $(u(.), w(.))$, if and only if $(u_1(.), w_1(.))$ are OLN equilibrium actions for the game defined by

$$\begin{aligned}\dot{x}_z(t) &= \begin{bmatrix} J - \frac{1}{2}\theta I & A_1 \\ 0 & D_1 \end{bmatrix} x_z(t) + \begin{bmatrix} 0 \\ B_{z_1}\end{bmatrix} u_1(t) + \begin{bmatrix} B_2 \\ 0 \end{bmatrix} w_1(t) \\ &=: A_zx_z(t) + B_{z1}u_1(t) + B_{z2}w_1(t), \end{aligned} \tag{30}$$

where $x_z^T(0) := x_{z0} = [([I\ 0]X^{-1}x_0)^T\ u_1^T(0)\ \cdots u_1^{(k-2)^T}(0)]$ is such that (28 and 29) hold, and

$$J_\gamma = \int_0^{t_f} \left\{z^T(t)L^T\bar{M}_\gamma Lz(t)\right\} \mathrm{d}t + z^T(t_f)L_1^T\bar{Q}_{t_f}L_1z(t_f). \tag{31}$$

Similar to the index-one case, to avoid including controls in the scrap value, we make the standard assumption that

$$L_1^T \bar{Q}_{t_f} L_1 = \begin{bmatrix} Q_{zt_f} & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{where } Q_{zt_f} \in \mathbb{R}^{n+(k-1)m \times n+(k-1)m}.$$

Moreover, let

$$L^T \bar{M}_\gamma L =: M_{z\gamma} =: \begin{bmatrix} Q_z & V_z & W_z \\ V_z^T & R_{\overline{11}z} & N_z \\ W_z^T & N_z^T & R_{\overline{22}\gamma z} \end{bmatrix}, \tag{32}$$

where $Q_z \in \mathbb{R}^{n+(k-1)m \times n+(k-1)m}$, $R_{jz} \in \mathbb{R}^{m_j \times m_j}$. Then, we can rewrite (31) as

$$J_\gamma = \int_0^{t_f} \left\{[z^T(t) M_{z\gamma} z(t)\right\} \mathrm{d}t + x_z^T(t_f) Q_{zt_f} x_z(t_f). \tag{33}$$

Next, assume that the set of admissible control functions U_s contains functions $u(.)$ that are $k-1$ times differentiable, and such that $u^{(k-1)}$ is piecewise continuous. The results from Sects. 3 and 4 can then be directly used to formulate robustness results for this higher-order index case. By replacing (7 and 12) with (30 and 32,33) (and deleting the scrap value for the infinite-horizon case), and using the corresponding notations, we can straightforwardly derive these equivalents.

7 Conclusions

In this paper, we considered the robust optimal control problem for descriptor systems, for finite and infinite planning horizons. We solved the problem assuming an open-loop information structure. We showed that RDE (9) and the critical value γ^* are critical to the finite planning horizon, and that ARE (17) and the critical value γ_{OL}^∞ are similarly critical to the infinite planning horizon. We also characterized the worst-case controller whether it can be synthesized as a feedback controller. First, we presented results for index-one systems and then showed how they can be applied to higher-order systems. We assumed that the cost function includes derivatives of the input, up to the index of the system.

In future research, we should investigate whether index reduction algorithms that were developed to reduce higher-order index systems to index-one systems (see e.g., [23] and [24]) can be used to solve the most general problem. If, for example, the corresponding disturbance attenuation problem can be similarly transformed into a disturbance attenuation problem for a system of index one, then the theory in this paper may be used to address robustness issues for such higher-order index systems.

It would also be interesting to consider other information structures such as delayed systems. Another topic related to this issue that needs further attention is the effect of the assumed information structure on the performance of the system. It is typically expensive to get more information on the system for control purposes. Therefore, it would be useful to compare the performance of a system operating under a robust controller using limited information, with a system operating under a controller using more advanced (costly) information. This may lead to a better understanding of the appropriate control for the system operating conditions.

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Appendix: State Transformation and Shorthand Notation (7, 8)

With X and Y defined as in (2), $X =: \left[\, X_1 \;\, X_2 \,\right]$ and $Y^T =: \left[\, Y_1^T \;\, Y_2^T \,\right]$ are nonsingular matrices. With the state transformation $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} := X^{-1}x(t)$, the corresponding state and control matrices for the reduced dynamical system are $A := J;\ B_1 := Y_1 B_u$ and $B_2 := Y_1 B_w$.

The matrices used in Eqs. (7) and (8) are $C_1 X =: \bar{C}_1 =: \left[\, \bar{C}_{11} \;\, \bar{C}_{12} \,\right]$, $C_2 X =: \bar{C}_2 =: \left[\, \bar{C}_{21} \;\, \bar{C}_{22} \,\right]$, $Q := X_1^T \bar{Q} X_1$, $V := -X_1^T \bar{Q} X_2 Y_2 B_u$, $N := B_u^T Y_2^T X_2^T \bar{Q} X_2 Y_2 B_w$, $W := -X_1^T \bar{Q} X_2 Y_2 B_w$, $R_{\bar{1}\bar{1}} := B_u^T Y_2^T X_2^T \bar{Q} X_2 Y_2 B_u + \bar{R}_1$, and $R_{\overline{22}\gamma} := B_w^T Y_2^T X_2^T \bar{Q} X_2 Y_2 B_w - \gamma \bar{R}_2$.

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