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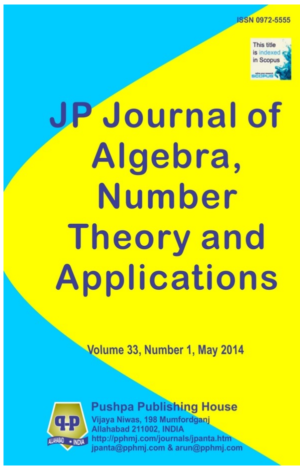


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# ON PRIMENESS OF PATH ALGEBRAS OVER A UNITAL COMMUTATIVE RING

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## Abstract

In this paper, we first discuss the primeness of basic ideals in a free $R$-algebra where $R$ is a unital commutative ring. The condition of primeness is applied to show a prime basic ideal in a path algebra $RE$ on a graph $E$. For every hereditary subset $H$, we can construct a (graded) basic ideal $I_H$ in $RE$. The basic ideal $I_H$ is an ideal of linear combinations of vertices in $H$ and paths whose ranges in $H$. The main purpose of this paper is to present the necessary and sufficient conditions on a graph, so that $I_H$ is a prime basic ideal, if $H$ is

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saturated hereditary. Since $\varnothing$ is saturated hereditary, we find the necessary and sufficient conditions on a graph, so that a path algebra *RE* is basically prime.

## 1. Introduction

Given a field *K* and a (directed) graph $E = (E^0, E^1, s, r)$, where $E^0$ is a set of vertices, $E^1$ is a set of edges and two functions $s, r : E^1 \to E^0$. A path algebra *KE* is a free *K*-algebra having a basis $E^*$ in which $E^*$ is a set of all paths in the graph [4, 6, 7]. Consider that the path algebra $KE = \underset{m \geq 0}{\oplus} KE^m$ is a graded algebra [6, 7] and an associative algebra [7]. Moreover, *KE* is a unital algebra if $E^0$ is finite and it has a finite dimension if *E* is an acyclic finite graph [7]. In this paper, we discuss a path algebra over a unital commutative ring as a generalization of the path algebra *KE*.

The path algebra *KE* can be extended to a Leavitt path algebra $L_K(E)$ over a field *K* on the extended graph with two conditions of Cuntz Krieger. It is important to note that $L_K(E)$ is also a free *K*-algebra [1, 2-6]. Furthermore, Tomforde [11] generalized $L_K(E)$ into $L_R(E)$ that is a Leavitt path algebra over a unital commutative ring *R*. He introduced the term of a basic ideal in $L_R(E)$ to define basically simple $L_R(E)$ [11]. Based on [11], Wardati et al. developed a definition of prime basic ideal in $L_R(E)$ to define basically prime $L_R(E)$ [12]. A generalization of a basic ideal in a free *R*-algebra over a unital commutative ring *R* has been discussed in [13]. A part of this paper will be devoted to the discussion of the primeness of the free *R*-algebra and its properties characterized by the prime basic ideals. This topic is a continuation of the paper [13] and it also refers to [8, 10, 14].

Larki [9] and Pino et al. [2, 5] defined that a nonempty subset $M \subseteq E^0$ is called a *maximal tail* if it satisfies three conditions *MT*1, *MT*2, *MT*3. They found a relationship between the conditions *MT*1, *MT*2 and the saturated hereditary nature. A subset $H \subset E^0$ is saturated hereditary if and only if $M = E^0 \backslash H$ satisfies $MT1$, $MT2$. Based on this property, $\varnothing$ is saturated hereditary and $E^0$ always satisfies the conditions $MT1$, $MT2$. The most important result is the necessary and sufficient condition of primeness of $L_K(E)$, i.e., $E^0$ is a maximal tail. In other words, $L_K(E)$ is prime if and only if $E^0$ satisfies the condition $MT3$.

The primeness of $L_K(E)$ viewed as a ring (an algebra over itself) is a result of Larki [9], that is $L_R(E)$ is prime if and only if $R$ is an integral domain and $E^0$ satisfies the condition $MT3$. Different from Larki, the primeness of $L_K(E)$ is a consequence of a proposition of Pino et al. in [2, 5]. They stated that if $H$ is saturated hereditary, then the graded ideal in $L_K(E)$ constructed by $H$, $I_H = Span_K\{\alpha\beta^* : \alpha, \beta \in E^*, r(\alpha) = r(\beta) \in H\}$ is a prime basic ideal if and only if $M = E^0 \backslash H$ is a maximal tail [2, 5]. Since $\varnothing$ is saturated hereditary and $I_\varnothing = \{0\}$, $L_K(E)$ is a prime algebra if and only if $I_\varnothing$ is a prime basic ideal if and only if $E^0$ is a maximal tail.

Based on [7], we can define an arrow ideal $I_E$ in $RE$, where $E$ is a connected finite graph. The arrow ideal $I_E$ is an ideal consisting of the linear combinations of paths of length $l \geq 1$. In other words, $I_E = Span_R\{\mu \in E^* \backslash E^0\}$ is a basic ideal that does not contain the vertices and it is only defined on the connected graph. If given any finite graph having an isolated vertex $u \in E^0$, i.e., $r^{-1}(u) = s^{-1}(u) = \varnothing$, then we can form a basic ideal $Ru$ that consists a linear combination of $u$. If a hereditary subset $H \subseteq E^0$ contains the isolated vertices, then $Span_R\{u : r^{-1}(u) = s^{-1}(u) = \varnothing\}$ $\oplus\, Span_R\{\mu \in E^* \backslash E^0 : r(\mu) \in H\}$ is also a basic ideal. It is clear that the last two basic ideals do not contain $H$. Furthermore, we can define a basic ideal constructed by the hereditary subset $H$, i.e., $I_H = Span_R\{\mu \in$

$E^* : r(\mu) \in H \vee \mu \in H\}$ that contains *H*. The basic ideal $I_H$ in *RE* plays an important role to discuss basically prime path algebra *RE*.

While the Leavitt path algebra $L_K(E)$ is always semiprime [3, 4, 6], the case for the path algebra *KE* is somewhat different. Pino et al. in [4, 6] showed that a necessary and sufficient condition of a semiprime path algebra *KE* is:

$$\text{for every } \mu \in E^*, \text{ there exists } v \in E^* \text{ such that}$$

$$r(\mu) = s(v),\ s(\mu) = r(v). \tag{1}$$

If (1) is not met, then *KE* is neither a semiprime nor a prime algebra. So (1) is a necessary condition of a prime path algebra *KE*. This indicates that (1) is also a necessary condition for a basically prime path algebra *RE*.

The main purpose of this paper is to determine a necessary and sufficient condition for the path algebra *RE* over a unital commutative ring is basically prime. We can show that if *H* is saturated hereditary and a basic ideal $I_H = Span_R\{\mu \in E^* : r(\mu) \in H \vee \mu \in H\}$ is prime, then $M = E^0 \backslash H$ is a maximal tail, but not the converse. The main result is if *H* is saturated hereditary, then $I_H$ is a prime basic ideal if and only if $M = E^0 \backslash H$ is a maximal tail and for every path *a* whose range $r(a) \in M$, there exists a path *b* such that $r(a) = s(b)$ and $s(a) = r(b)$. Furthermore, the path algebra *RE* is basically prime if and only if for every $v, w \in E^0$ there exists $y \in E^0$ such that $v \leq y, w \leq y$ and for every path $\mu$, there is a path $\nu$ such that $r(\mu) = s(\nu)$ and $s(\mu) = r(\nu)$.

## 2. Basic Properties of Graphs

The discussions on the path algebras over a field can be found in [4, 6, 7]. We first recall the notion of a quiver or directed graph and its properties to discuss path algebras over a unital commutative ring. In further discussion, a directed graph is stated by a graph only.

**Definition 2.1** [7]**.** A graph $E = (E^0, E^1, s, r)$ is 4-tuples consisting of two sets $E^0$ (whose elements are called *vertices*) and $E^1$ (whose elements are called *edges*) and two maps $s, r : E^1 \to E^0$. For every edge $e \in E^1$, its source $s(e)$ and its range $r(e)$ are in $E^0$.

A graph $E$ is called *row-finite graph* if $s^{-1}(v)$ is finite for every $v \in E^0$. A vertex $u \in E^0$ is called an *isolated vertex* if $r^{-1}(u) = s^{-1}(u) = \varnothing$, i.e., $r(e) \neq u$, $s(e) \neq u$ for every $e \in E^1$. In this paper, we restrict to a finite graph and row-finite graph. Throughout we simply write a finite graph.

A path $\mu = e_1e_2 \ldots e_m$ of length $m \geq 1$ on a graph $E$ is a sequence of edges such that $r(e_i) = s(e_{i+1})$ with $e_i \in E^1$, $i = 1, 2, ..., m-1$, where source and range of $\mu$ are $s(\mu) = s(e_1)$, $r(\mu) = r(e_m)$, respectively. The path $\mu = e_1e_2 \ldots e_m$ is called *cycle* if $r(\mu) = s(\mu)$ and $s(e_i) \neq s(e_j)$ for every $i \neq j$. A cycle of length 1 is called a *loop*. A set of all paths of length $n$ is denoted as $E^n$, so that every vertex is a path of length 0. Furthermore, $E^*$ denote a set of all paths in graph $E$. The composition of any two paths can be defined as a *multiplicative operation* on $E^*$. Then the multiplication of $\mu = \mu_1 \ldots \mu_l$ and $v = v_1 \ldots v_k$ is defined as:

$$\mu v = \begin{cases} \mu_1 \ldots \mu_l v_1 \ldots v_k, & \text{if } r(\mu_l) = s(v_1), \\ 0, & \text{if } r(\mu_l) \neq s(v_1). \end{cases} \tag{2}$$

This refers to [7] and we use it to define a path algebra over a unital commutative ring.

**Definition 2.2.** Given a unital commutative ring $R$ and a graph $E = (E^0, E^1, s, r)$. A path algebra over $R$ on the graph $E$ denoted $RE$ is a free $R$-algebra whose a basis $E^*$ such that the multiplication of two basis vectors defined by (2) satisfies two conditions: $v_i v_j = \delta_{ij} v_i$ and $e_i = e_i r(e_i) = s(e_i) e_i$, for every $v_i, v_j \in E^0$, $e_i \in E^1$.

The first condition in Definition 2.2 shows that every vertex in the graph is an idempotent element and any two distinct vertices form a pair of orthogonal idempotent elements in *RE*. Moreover, based on Definition 2.2, the path algebra *RE* has the properties stated in the following lemma.

**Lemma 2.3.** *Let RE be a path algebra over a unital commutative ring on a graph E. Then the following statements apply*:

(1) *RE is a graded associative algebra*.

(2) *RE has an identity if and only if* $E^0$ *is finite*.

(3) *RE has a finite dimension if and only if E is a finite acyclic graph*.

**Proof.** Based on Definition 2.2, it is easy to show that *RE* is a graded algebra, i.e., $RE = \bigoplus_{m \geq 0} RE^m$, where $RE^m$ is an *R*-submodule of *RE* for every $m \geq 0$ and every nonnegative integer $k, l, (RE^k)(RE^l) \subseteq RE^{k+l}$. A proof of an associative algebra *RE*, points (2) and (3) is analogous to the proof of [7]. □

An ideal in *RE* can be constructed by a hereditary subset of $E^0$. The definition of the hereditary subset is related to a preorder relation $\leq$ on the vertex set $E^0$ defined by Abrams and Pino [1] as: for every $v, w \in E^0$, $v \leq w$ if and only if $v = w$ or there is a path $\mu$ such that $s(\mu) = v$, $r(\mu) = w$. According to [1], a subset $H \subset E^0$ is hereditary if $v \in H$ and $v \leq w$ imply $w \in H$. Then the subset *H* is called *saturated* if $s^{-1}(v) \neq \varnothing$ and $r(s^{-1}(v)) \subseteq H$ imply $v \in H$.

We know that the tree of $v \in E^0$ is defined and denoted as $T(v) = \{w \in E^0 : v \leq w\}$. This definition can be extended to the tree of a subset

$X \subset E^0$, namely $T(X) = \bigcup_{v \in X} T(v)$. For every $v \in X$, $v \in T(v)$, $X \subseteq T(X)$. Therefore, saturated hereditary closure of $X$ denoted $\overline{X}$ is the smallest saturated hereditary subset that contains $X$. This means that for every saturated hereditary subset $H \supseteq X$ implies $\overline{X} \subseteq H$. To determine $\overline{X}$ requires the following lemma which has been proven in [2].

**Lemma 2.4.** *Given a graph* $E$ *and* $X \subseteq E^0$. *Then* $\overline{X} = \bigcup_{n=0}^{\infty} G_n(X)$, *where* $G_0(X) = T(X)$ *and*

$$G_n(X) = \{v \in E^0 : s^{-1}(v) \neq \emptyset,\ r(s^{-1}(v)) \subseteq G_{n-1}(X)\} \cup G_{n-1}(X)$$

*for every* $n \geq 1$.

Besides the saturated hereditary subset, there is a nonempty subset of $E^0$ that meets the conditions of maximal tail. According to [2, 5, 9], the maximal tail is defined as follows:

**Definition 2.5.** Let $E$ is a graph. Then a nonempty subset $M \subseteq E^0$ is called a *maximal tail* if it implies:

$MT1$. If $v \in E^0$, $w \in M$ and $v \leq w$, then $v \in M$.

$MT2$. If $v \in M$ with $s^{-1}(v) \neq \emptyset$, then there is $e \in E^1$ with $s(e) = v$ and $r(e) \in M$.

$MT3$. For every $v, w \in M$, there is $y \in M$ such that $v \leq y$ and $w \leq y$.

The saturated hereditary subset is interrelated to some conditions of maximal tail. The relationship is stated in the following lemma that has been proven by [2].

**Lemma 2.6.** *Given a graph* $E$ *and* $H \subset E^0$. *Then* $H$ *is saturated hereditary if and only if* $M = E^0 \backslash H$ *satisfies the conditions* $MT1$, $MT2$.

## 3. The Primeness of Free Algebras over a Unital Commutative Ring

This section is a continuation of the paper [13], so we refer some of the previous results. The algebra considered in this section is always a unital free *R*-algebra that is simply written as a free *R*-algebra. We first recall the definition of an ideal in the free *R*-algebra that meets a certain requirement.

**Definition 3.1** [13]**.** Let *A* be a free *R*-algebra with a basis *X*. Then an ideal *I* in *A* is called a *basic ideal* if $kx \in I$ for every non-zero $k \in R,$ and every $x \in X,$ implies $x \in I.$

In summary, the properties are stated in the following proposition (see Proposition 2.7 and Lemma 2.8 in [13]).

**Proposition 3.2.** *Let A be a free R-algebra with a basis X and I be an ideal of A. Then we have the following assertions*:

(1) *I is a basic ideal if and only if I is a free ideal, namely the ideal I has a basis in X.*

(2) *If* $h \in X,$ *then* $(h) = \left\{ \sum_i a_i h b_i : a_i, b_i \in A \right\}$ *is a basic ideal.*

The basic ideal is very important to discuss the primeness of a free *R*-algebra. Analogous to the special properties of an ideal in a ring [10, 14], we can define a prime basic ideal. There is a class of the free *R*-algebras characterized by the basic ideal.

**Definition 3.3.** Let *A* be a free *R*-algebra and *I* be a basic ideal of *A*.

(1) *I* is a prime basic ideal whenever $I \neq A$ and any two basic ideals $P, Q \subseteq A,$ if $PQ \subseteq I,$ then $P \subseteq I$ or $Q \subseteq I.$

(2) The algebra *A* is called *basically prime algebra* if the zero ideal is the prime basic ideal.

The definition of basically prime algebra is based on the primeness of zero basic ideal. We need the properties of prime basic ideal to discuss the basically prime algebra. This is in line to the one presented in [10] and [14].

**Proposition 3.4.** *Let $A$ be a free $R$-algebra with a basis $X$ and $P$ be a basic ideal of $A$. Then the following assertions are equivalent*:

(1) *$P$ is a prime basic ideal*.

(2) *For every $a, b \in A$, if $(a), (b)$ are basic ideals, and $(a)(b) \subseteq P$, then $a \in P$ or $b \in P$.*

(3) *For every $a, b \in A$, if $(a), (b)$ are basic ideals, and $aAb \subseteq P$, then $a \in P$ or $b \in P$.*

**Proof.** It is clear to prove (1) $\Rightarrow$ (2) $\Rightarrow$ (3). To prove (3) $\Rightarrow$ (1), take any two basic ideals $I, J \subseteq A$ such that $IJ \subseteq P$ but $I \not\subseteq P$. According to Proposition 3.2 point (1), the basic ideals $I$, $J$ are free ideals. Then there is a basis vector $a \in I$, $a \notin P$. Suppose $\{b_1, ..., b_m\}$ is a basis of $J$. According to Proposition 3.2 point (2), $(a)$, $(b_i)$ are basic ideals for any $i$, $1 \leq i \leq m$. In addition, $aAb_i \subseteq IJ \subseteq P$ and based on (3), we find $b_i \in P$. For any $x \in J$, $x = \sum_{i=1}^{m} r_i b_i$ for some $r_i \in R$ and all basis vectors $b_i \in J$, $1 \leq i \leq m$, then $x = \sum_{i=1}^{m} r_i b_i \in P$. It means that $J \subseteq P$. Hence, any two basic ideals $I, J \subseteq A$ such that $IJ \subseteq P$ and $I \not\subseteq P$, then $J \subseteq P$. Similarly to show that any two basic ideals $I, J \subseteq A$ such that $IJ \subseteq P$ and $J \not\subseteq P$, then $I \subseteq P$. Thus, the basic ideal $P$ is prime. □

A basically prime free $R$-algebra $A$ depends on the primeness of the zero basic ideal. Based on Definition 3.3 and Proposition 3.4, $\{0\}$ is a prime basic ideal if and only if for every $a, b \in A$, if $(a)$, $(b)$ are basic ideals and $(a)(b) = \{0\}$, then $a = 0$ or $b = 0$, if and only if for every $a, b \in A$, if $(a)$, $(b)$ are basic ideals and $aAb = \{0\}$, then $a = 0$ or $b = 0$. Furthermore, if $X$ is a basis of $A$, then $\{0\} \neq (x)$ is a basic ideal for every $x \in X$ (Proposition 3.2, point (2)). In addition, each element in $A$ is a linear combination of the elements of $X$. Then we have a corollary stated as follows:

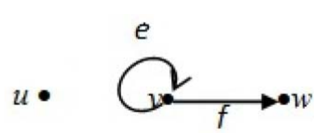
**Corollary 3.5.** *Let A be a free R-algebra with a basis X. Then A is basically prime algebra if and only if* $(x)(y) \neq \{0\}$ *if and only if* $xAy \neq \{0\}$ *for every* $x, y \in X$.

## 4. Prime Basic Ideals in Path Algebras over a Ring

Throughout this section, the path algebra *RE* is a free *R*-algebra on a finite graph *E*, where $E^*$ is a basis of *RE*. The arrow ideal $I_E = \underset{l \geq 1}{\oplus} KE^l$ in path algebras over a field studied in [7], is an ideal generated by $E^* \backslash E^0$, where *E* is a connected finite graph. Similarly, we define $I(E) = \underset{l \geq 1}{\oplus} RE^l$ is an arrow ideal in *RE*. Consequently, the ideal $I(E)$ is an ideal that does not contain a vertex and it is only defined on a connected finite graph *E*. Analogous to the definition of the arrow ideal, when given a finite graph *E* and $\varnothing \neq X \subseteq E^0$, where each vertex is not isolated and there is an edge $e \in E^1$ such that $r(e) \in X$. Then we can define the set of all paths whose ranges in *X*, as follows:

**Definition 4.1.** Given a graph *E* and $\varnothing \neq X \subseteq E^0$ such that every vertex in *X* is not isolated and there is an edge $e \in E^1$ such that $r(e) \in X$. We can define a set of all linear combinations of paths whose ranges in *X* and they are not vertices, i.e., $Span_R\{\mu \in E^* \backslash E^0 : r(\mu) \in X\}$.

It is easy to show that $Span_R\{\mu \in E^* \backslash E^0 : r(\mu) \in X\}$ is an *R*-submodule of *RE*, but it is not necessarily an ideal. Note the following graph *G* with $G^0 = \{u, v, w\}$ and $G^1 = \{e, f\}$:



We have a path algebra $RG = \langle\{u, v, w, e, f, e^2, ef, e^3, e^2f, ..., e^m, e^{m-1}f, ...\}\rangle$. If $X = \{v\}$, then $Span_R\{\mu \in E^*\backslash E^0 : r(\mu) = v\} = \langle\{e, e^2, ..., e^n, ...\}\rangle$ is an $R$-submodule but not an ideal in $RG$.

**Proposition 4.2.** *Given a graph $E$ and $X \subset E^0$ such that every vertex in $X$ is not isolated and there is an edge $e \in E^1$ such that $r(e) \in X$. Then $Span_R\{\mu \in E^*\backslash E^0 : r(\mu) \in X\}$ is a graded basic ideal if and only if $X$ is hereditary.*

**Proof.** Consider $J = Span_R\{\mu \in E^*\backslash E^0 : r(\mu) \in X\}$. We first suppose that $X$ is not hereditary. Take any $a = \sum_{k_i \in R} k_i\mu_i \in J$ and a monomial $cx \in RE$ such that $r(x) \notin X$ and $0 \neq a(cx) = \left(\sum_{k_i \in R} k_i\mu_i\right)cx = \sum_{k_i \in R} k_ic(\mu_ix).$ Then there is a monomial $k_lc(\mu_lx) \neq 0$ such that $s(x) = r(\mu_l) \in X$ but $r(\mu_lx) = r(x) \notin X$. We find $k_lc(\mu_lx) \notin J$, so $J$ is not an ideal and there is a contradiction. Hence, $X$ should be hereditary. For the converse, take any $a, b \in J$ and any monomial $kx \in RE$ with $k \in K$, $x \in E^*$. Then we have $a = \sum_{k_i \in R} k_i\mu_i$, $b = \sum_{k_j \in R} k_j\mu_j$ for some $\mu_i, \mu_j \in E^*\backslash E^0$, $r(\mu_i), r(\mu_j) \in X$. It is clear that $a - b \in J$. Furthermore, for every $\mu_i \in J$, $r(\mu_i) \neq s(x)$, we have $a(kx) = \left(\sum_{k_i \in R} k_i\mu_i\right)(kx) = \left(\sum_{k_i \in R} k_ik(\mu_ix)\right) = 0 \in J$. Likewise, if there is $\mu_i \in J$ with $s(x) = r(\mu_i) \in X$, then $\mu_ix \neq 0$, and since $X$ is *hereditary*, $r(\mu_ix) = r(x) \in X$, so that $0 \neq a(kx) = \left(\sum_{k_i \in R} k_ik(\mu_ix)\right) \in J$. Hence, $a(kx) \in J$. Similarly, we get $(kx)a \in J$. In other words, $J$ is an ideal of $RE$. Based on Definition 4.1, $J$ is a free ideal, so that $J$ is a basic ideal. Since

$J \cap RE^0 = \{0\}$ and

$$J = Span_R\{\mu \in E^* \backslash E^0 : r(\mu) \in X\}$$

$$= \underset{l \geq 1}{\oplus} Span_R\{\mu \in E^* \backslash E^0 : r(\mu) \in X, |\mu| = l\},$$

$J = \underset{k \geq 0}{\oplus} J_k$, where $J_k = J \cap RE^k$ is an $R$-submodule of $RE$. Furthermore, it is easy to show that $J_l J_m \subseteq J_{l+m}$. Hence, $J$ is a graded basic ideal. □

If the finite graph $E$ is connected, then the all vertices are not isolated. It means that Proposition 4.2 implies the following corollary.

**Corollary 4.3.** *If the finite graph $E$ is connected, then the arrow ideal $I_E$ is a graded basic ideal in $RE$.*

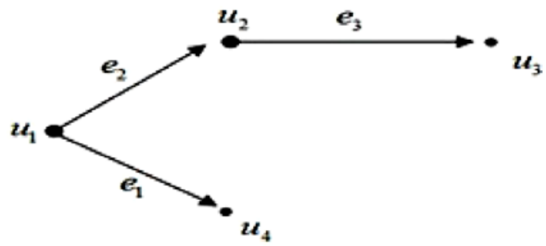
A hereditary subset $H \subseteq E^0$ may contain an isolated vertex. On the graph $G$, $H = \{u, w\}$ is hereditary subset containing an isolated vertex $u$ then $Span_R\{u\} = \langle u \rangle$ is a basic ideal. In addition, we can form a basic ideal generated by the isolated vertex and the paths that are not vertices and the ranges of the paths equal to $w$, i.e.,

$$Span_R\{u\} \oplus Span_R\{\mu \in E^* \backslash E^0 : r(\mu) = w \in H\} = \langle\{u, f, ef, ..., e^n f, ...\}\rangle.$$

Analogous to the proof of Proposition 4.2, the last basic ideal is graded but does not contain $H$. This inspires us to define a graded basic ideal which is constructed by a hereditary subset contained in the ideal.

**Definition 4.4.** Given a finite graph $E$ and a hereditary subset $H \subseteq E^0$. We define a graded basic ideal $I_H = Span_R\{\mu \in E^* : r(\mu) \in H \vee \mu \in H\}$.

Consider a graph $Q$ with $Q^0 = \{u_1, u_2, u_3, u_4\}$ and $Q^1 = \{e_1, e_2, e_3\}$ as follows:




The acyclic graph $Q$ does not contain an isolated vertex. We have a free $R$-algebra $RQ$ with a finite basis $\{u_1, u_2, u_3, u_4, e_1, e_2, e_3, e_2e_3\}$, and the arrow ideal $I(Q) = \langle\{e_1, e_2, e_3, e_2e_3\}\rangle$. Furthermore, $H_1 = \{u_3\}$ is hereditary but not saturated and $\overline{H}_1 = \{u_2, u_3\}$ is a saturated hereditary closure of $H_1$. Then $I_{H_1} = \langle\{u_3, e_3, e_2e_3\}\rangle \subsetneq I_{\overline{H}_1} = \langle\{u_2, u_3, e_2, e_3, e_2e_3\}\rangle$. From the graph $G$, we know that $H = \{u, w\}$, $K = \{v, w\}$ and $H \cap K$ are hereditary subsets. Then we have $I_H = \langle\{u, w, f, ..., e^n f, ...\}\rangle$, $I_K = \langle\{v, w, e, f, ..., e^{n-1} f, e^n, ...\}\rangle$, $I_{H\cap K} = \langle\{w, f, ef, e^2 f, ..., e^n f, ...\}\rangle$ and $I_K I_H = \langle\{w, f, ef, ..., e^n f, ...\}\rangle =$ $I_{H\cap K} = I_H I_K$. The above cases illustrate the properties of basic ideals constructed by a hereditary subset.

**Proposition 4.5.** *Given a finite graph $E$ and the hereditary subsets $H_1, H_2 \subseteq E^0$. Then we have the following*:

(1) *If* $H_1 \subseteq H_2$, *then* $I_{H_1} \subseteq I_{H_2}$.

(2) $I_{H_1} I_{H_2} = I_{H_1 \cap H_2}$.

**Proof.** We can prove (1) directly from the definition. Furthermore, take any $x \in I_{H_1} I_{H_2}$, then $x = ab$ such that

$$a = \sum_{\substack{c_i \in R \\ u_i \in H_1}} c_i u_i + \sum_{k_i \in R} k_i a_i \in I_{H_1}, \quad b = \sum_{\substack{d_j \in R \\ v_j \in H_2}} d_j v_j + \sum_{l_j \in R} l_j b_j \in I_{H_2}$$

with $a_i, b_j \in E^*\backslash E^0$, and $r(a_i) \in H_1$, $r(b_j) \in H_2$. Then we have

$$x = \sum_{\substack{c \in R \\ w \in H_1 \cap H_2}} cw + \sum_{k_i \in R} \sum_{l_j \in R} k_i l_j a_i b_j \in I_{H_1 \cap H_2}.$$

If $x \neq 0$, then there exist $w \in H_1 \cap H_2$ or $a_i, b_j \in E^*\backslash E^0$ with $r(a_i) = s(b_j) \in H_1$ such that $a_i b_j \in E^*\backslash E^0$, where $r(a_i b_j) = r(b_j) \in H_2$ and since $s(b_j) \in H_1$ and $H_1$ is hereditary, so that $r(a_i b_j) \in H_1$. We find $r(a_i b_j) \in H_1 \cap H_2$ or $a_i b_j \in I_{H_1 \cap H_2}$. Hence, $I_{H_1} I_{H_2} \subseteq I_{H_1 \cap H_2}$. For the converse, take any $y \in I_{H_1 \cap H_2}$. Then $y = \sum_{k_i \in R} k_i w_i + \sum_{c_j \in R} c_j \mu_j$ for some $w_i \in H_1 \cap H_2 \subseteq H_1$, $\mu_j \in E^*\backslash E^0$ with $r(\mu_j) \in H_1 \cap H_2 \subseteq H_1$, so that $y \in I_{H_1}$. Since $E^0$ is finite, $H_1 \cap H_2$ is also finite. Suppose $\{w_1, ..., w_m\} = H_1 \cap H_2 \subseteq H_2$ and $c = \sum_{i=1}^{m} w_i$, $c \in I_{H_2}$. For every $i$, $j$, $w_i$ is idempotent and $\mu_j r(\mu_j) = \mu_j$, then we have

$$yc = \left( \underbrace{\sum_{k_i \in R} k_i w_i + \sum_{c_j \in R} c_j \mu_j}_{\in I_{H_1}} \right) \left( \underbrace{\sum_{i=1}^{m} w_i}_{\in I_{H_2}} \right) = \left( \sum_{k_i \in R} k_i w_i + \sum_{c_j \in R} c_j \mu_j \right) = y.$$

Hence, $y \in I_{H_1} I_{H_2}$. □

If we reexamine the previous two graphs, $H$, $H \cap K$ are the saturated hereditary subsets in $G^0$, as well as $\overline{H}_1$ in $Q^0$. Based on Lemma 2.6, the complements of them are, respectively, $M$, $M'$ and $M_1'$ which satisfy two conditions $MT1$, $MT2$. However, only $M'$ does not meet $MT3$, so that both

$M$, $M_1'$ are maximal tails. According to Proposition 3.4, we can investigate that $I_{H \cap K}$, $I_{\overline{H}_1}$ are not prime basic ideals and the only prime basic ideal is $I_H$.

If $H$ is a saturated hereditary subset of $E^0$, then we have a necessary condition of primeness of the basic ideal $I_H$, i.e., $M = E^0 \backslash H$ is a maximal tail. The next theorem states the necessary and sufficient conditions of primeness of a basic ideal $I_H$. This theorem is an important result in this paper.

**Theorem 4.6.** *Let E be a finite graph and subset $H \subset E^0$ be a saturated hereditary. A* (*graded*) *basic ideal $I_H$ is prime if and only if $M = E^0 \backslash H$ is a maximal tail and for every path* μ *whose a range $r(\mu)$ in M, there is a path* ν *in RE such that $r(\mu) = s(\nu)$ and $s(\mu) = r(\nu)$.*

**Proof.** Based on Lemma 2.6, since $H$ is saturated hereditary, $M = E^0 \backslash H$ meets the conditions *MT*1, *MT*2. Suppose $M$ does not meet *MT*3, namely,

$$\text{there exist } v,\, w \in M \text{ such that } v \not\leq y \text{ or } w \not\leq y \text{ for every } y \in M. \tag{3}$$

According to Lemma 2.4, we have $\overline{T(v)} = \bigcup_{i=0}^{n} G_i(T(v))$, where $G_0(T(v)) = T(v)$ and $G_i(T(v)) = \{x \in E^0 : 0 \neq r(s^{-1}(x)) \subseteq G_{i-1}(T(v))\} \cup G_{i-1}(T(v))$ for every $i \geq 1$. We would show that $\overline{T(v)} \cap \overline{T(w)} \cap M = \varnothing$ as follows: suppose $\overline{T(v)} \cap \overline{T(w)} \cap M \neq \varnothing$. Then there is the smallest integer $k$, $0 \leq k \leq n$ such that $G_k(T(v)) \cap \overline{T(w)} \cap M \neq \varnothing$. If $0 < k \leq n$, then there exists $x \in G_k(T(v)) \cap \overline{T(w)} \cap M$ and $G_{k-1}(T(v)) \cap \overline{T(w)} \cap M = \varnothing$. On the other hand, $\overline{T(w)}$ is hereditary, then $\varnothing \neq r(s^{-1}(x)) \subseteq G_{k-1}(T(v)) \cap \overline{T(w)} \subseteq H$. Since $H$ is hereditary, we find $x \in H$ which contradicts to $x \in M$. If $k = 0$ or

$T(v) \cap \overline{T(w)} \cap M \neq \varnothing$, then there is the smallest integer $l, 0 \leq l \leq n$ such that $T(v) \cap G_l(T(w)) \cap M \neq \varnothing$. Since $T(v)$ is hereditary, there is analogously a contradiction if $0 < l \leq n$. If $l = 0$, then $T(v) \cap T(w) \cap M \neq \varnothing$ or there exists $y \in T(v) \cap T(w) \cap M$ such that $v \leq y$, $w \leq y$ and $y \in M$. This contradicts to (3). Hence, we have $\overline{T(v)} \cap \overline{T(w)} \cap M = \varnothing$ or $\overline{T(v)} \cap \overline{T(w)} \subseteq H$. Then $I_{\overline{T(v)}} I_{\overline{T(w)}} \subseteq I_{\overline{T(v)} \cap \overline{T(w)}} \subseteq I_H$ because of Proposition 4.5. Since $I_H$ is a prime basic ideal, it implies $I_{\overline{T(v)}} \subseteq I_H$ or $I_{\overline{T(w)}} \subseteq I_H$. If $I_{\overline{T(v)}} \subseteq I_H$ and take any path $a \in I_{\overline{T(v)}} \subseteq I_H$ such that $r(a) = v$, then $v \in H$ which contradicts to $v \in M$. We have a contradiction similarly if $I_{\overline{T(w)}} \subseteq I_H$. Thus, $M = E^0 \backslash H$ meets (*MT*3). In other words, $M$ is a maximal tail. Furthermore, suppose that there is a path $a$ with $r(a) \in M$ such that for every path $b \in RE$ implies $r(a) \neq s(b)$ or $s(a) \neq r(b)$. Then $a(RE)a = 0 \subset I_H$. Since $I_H$ is a prime basic ideal, based on Proposition 3.4, $a \in I_H$. It means that $r(a) \in H$ which contradicts to $r(a) \in M$. Thus, for every path $\mu$ whose a range $r(\mu)$ in $M$, there is a path $\nu$ in $RE$ such that $r(\mu) = s(\nu)$ and $s(\mu) = r(\nu)$. The contrary proof, suppose that the basic ideal $I_H$ is not prime. Then there exist the paths $\alpha, \beta \in RE$ such that $(\alpha)$, $(\beta)$ are basic ideals, where $\alpha RE \beta \subseteq I_H$ but $\alpha, \beta \notin I_H$. If $\{0\} \neq \alpha RE \beta \subseteq I_H$, then there is a path $\delta \in RE$ such that $0 \neq \alpha\delta\beta \in I_H$. It means $r(\alpha\delta\beta) = r(\beta) \in H$ or $\beta \in I_H$. Then there is a contradiction. The next, if $\alpha RE \beta = \{0\}$, then for every path $\mu \in E^*$ such that $\alpha u \beta = 0$ with $\alpha, \beta \notin I_H$ has some possibilities. The first case, $\alpha, \beta \notin I_H$ with $\alpha, \beta \in E^* \backslash E^0$ or $\alpha, \beta \in E^0$, then $r(\alpha), r(\beta) \in M$ or $\alpha, \beta \in M$. Since $\alpha\mu = 0$ or $\mu\beta = 0$, we find $r(\alpha) \neq s(\mu)$ or $r(\mu) \neq s(\beta)$ which contradicts to the second condition of hypothesis. The second case, $\alpha, \beta \notin I_H$ with $\alpha \in E^* \backslash E^0$,

$\beta = u \in E^0$, then $r(\alpha)$, $u \in M$. If $\alpha\mu = 0$, then $r(\alpha) \neq s(\mu)$ that contradicts to the second condition. If $\alpha\mu \neq 0$ and $\mu\beta = \mu u = 0$, then we have $r(\alpha) = s(\mu) \in M$ and $r^{-1}(u) = \emptyset$, so that $s^{-1}(u) = \emptyset$ because of the following: suppose $s^{-1}(u) \neq \emptyset$. Then there is an edge $e \in E^1$ such that $u = s(e) = r(\alpha)$ which contradicts to $r^{-1}(u) = \emptyset$. Hence, the vertex $u \in M$ is isolated, so that $M$ is not a maximal tail. It contradicts to the first condition. The last case, $\alpha, \beta \notin I_H$ with $\alpha = u \in E^0$, $\beta \in E^* \backslash E^0$. The proof of this case is analogous to the second case. □

The trivial subset $\emptyset$ is hereditary and $\{0\} = I_{\emptyset}$, then according to Theorem 4.6, 0 is a prime basic ideal if and only if $M = E^0$ is a maximal tail and for every path $\mu$, there is a path $\nu$ such that $r(\mu) = s(\nu)$ and $s(\mu) = r(\nu)$. Then Theorem 4.6 implies the discovery of the necessary and sufficient conditions of basically prime path algebra $RE$ stated in the following corollary:

**Corollary 4.7.** *The path algebra RE on a finite graph E is basically prime if and only if for every* $v, w \in E^0$, *there exists* $y \in E^0$ *such that* $v \leq y$, $w \leq y$ *and for every path* $\mu$, *there is a path* $\nu$ *such that* $r(\mu) = s(\nu)$ *and* $s(\mu) = r(\nu)$.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