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The Open-Loop Zero-Sum Linear Quadratic Impulse Free Descriptor Differential Game

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ABSTRACT

In this paper we study the open-loop zero-sum linear quadratic differential game for descriptor systems that have index one. We present both necessary and sufficient conditions for the existence of an open-loop Nash equilibrium. We also relate the existence of an open-loop Nash equilibrium with the theory of invariant subspaces.

Keywords: zero-sum linear quadratic differential game, open-loop information structure, descriptor systems.

2000 Mathematics Subject Classification: 91A23, 91A05, 91A10.

1 INTRODUCTION

In the last decade significant progress has been made in the study of linear quadratic differential game. A linear quadratic differential game is a mathematical model that represents a conflict between different agents which control a dynamical system and each of them is trying to minimize his individual quadratic objective function by giving a control to the system. For this purpose, linear quadratic differential games have been applied in many different fields such as economic competitions among companies, environmental management games, armed conflicts, and parlor games (Haurie and Krawczyk, 2000).

Although the game has been applied in many fields, however, not all systems can be represented by an ordinary differential game. These situations occur when the systems are formulated as a set of coupled differential and algebraic equations. Descriptor systems can be used to model such systems and provide the ability to model more accurately the structure of physical systems, including non-dynamic modes and impulsive modes (Katayama and Minamino, 1992).

This paper is the continuation of the work of (Engwerda and Salmah, 2009) and (Musthofa, Salmah, Engwerda and Suparwanto, 2011) where the general linear quadratic differential game is considered for descriptor systems of index one. In this paper we elaborate the special important case when the game is zero-sum. We will show that under this extra simplifying condition it is possible to derive more explicit necessary and sufficient conditions for the existence of an open-loop Nash equilibrium both for a finite planning horizon and infinite planning horizon. We will also use some theory of invariant subspaces to present an existence result of open-loop Nash equilibria when the game is defined on an infinite planning horizon. We assume that the players act non-cooperatively and the only information they have is the present state and the model structure (Engwerda, 2005). In this paper we solve the problem by changing the descriptor differential game into ordinary differential game. A different approach for such problem has been done by (Salmah, 2009) and (Salmah, 2006) where the problem is solved directly without modifying into ordinary game.

This paper is going to be organized as follows; Section II will include some basic results of linear quadratic differential games for descriptor systems and also state the main purpose of this paper. Section III will present the main result for the zero-sum game on a finite planning horizon while Section IV will deal with an infinite planning horizon. In Section V we study the relationship between certain invariant subspaces and solutions of the algebraic Riccati equation. Section VI will illustrate some result in an examples from the previous sections. At last, section VII will conclude.

2 PRELIMINARIES

The game considered in this paper is a game described by the dynamical system

$$E\dot{x}\left(t\right)=Ax\left(t\right)+B_{1}u_{1}\left(t\right)+B_{2}u_{2}\left(t\right),\quad x\left(0\right)=x_{0} \tag{2.1}$$

where $E,A\in\mathbb{R}^{(n+r)\times(n+r)}$, $\operatorname{rank}\left(E\right)=n$, $B_i\in\mathbb{R}^{(n+r)\times m_i}$, $u_i\in\mathbb{R}^{m_i}$ are the actions player i can use to control the system and x_0 is the initial state. Each player has a quadratic cost functional J_i given by

$$\int_{0}^{t_f}\left\{x^{T}\left(t\right)\bar{Q}_{i}x\left(t\right)+u_{i}^{T}\left(t\right)\bar{R}_{i}u_{i}\left(t\right)\right\}dt+x^{T}\left(t_{f}\right)\bar{Q}_{it_f}x\left(t_{f}\right). \tag{2.2}$$

We start this section by stating some required basic results (Engwerda and Salmah, 2009). First, we recall some results from (Brenan, Campbell and Petzold, 1996) concerning the differential algebraic equation

$$E\dot{x}\left(t\right)=Ax\left(t\right)+f\left(t\right),\quad x\left(0\right)=x_{0} \tag{2.3}$$

and the associated matrix pencil

$$\lambda E-A. \tag{2.4}$$

System (2.3) and (2.4) are said to be regular if the characteristic polynomial $\det\left(\lambda E-A\right)$ is not identically zero. Then, from (Gantmacher, 1959) we recall the so-called Weierstrass canonical form.

Theorem 2.1. *If (2.4) is regular, then there exist nonsingular matrices X and Y such that*

$$Y^T E X = \begin{bmatrix} I_n & 0 \\ 0 & N \end{bmatrix} \text{ and } Y^T A X = \begin{bmatrix} J & 0 \\ 0 & I_r \end{bmatrix} \tag{2.5}$$

where J is a matrix in Jordan form whose elements are the finite eigenvalues, $I_k \in \mathbb{R}^{k\times k}$ is the identity matrix and N is a nilpotent matrix also in Jordan form. J and N are unique up to permutation of Jordan blocks.

If (2.4) is regular, then the solutions of (2.3) take the form ((Engwerda and Salmah, 2009), (Gantmacher, 1959))

$$x(t) = X_1 x_1(t) + X_2 x_2(t)$$

where with $X = \begin{bmatrix} X_1 & X_2 \end{bmatrix}$, $Y = \begin{bmatrix} Y_1^T & Y_2^T \end{bmatrix}$, $X_1, Y_1^T \in \mathbb{R}^{(n+r)\times n}$, $X_2, Y_2^T \in \mathbb{R}^{(n+r)\times r}$, and $x_1(t) = e^{Jt} x_1(0) + \int\limits_0^t e^{J(s-t)} Y_1 f(s)\, ds$, $x_1(0) = \begin{bmatrix} I_n & 0 \end{bmatrix} X^{-1} x_0$, $x_2(t) = -\sum\limits_{i=0}^{k-1} N^i Y_2 \frac{d^i}{dt^i} f(t)$, under the consistency condition :

$$\begin{bmatrix} 0 & I_r \end{bmatrix} X^{-1} x_0 = -\sum_{i=0}^{k-1} N^i Y_2 \frac{d^i}{dt^i} f(0).$$

Here k is the degree of nilpotency of N, that is the integer k for which $N^k = 0$ and $N^{k-1} \neq 0$. The index of the pencil (2.4) and of the descriptor system (2.3) is the degree k of nilpotency of N. If E is nonsingular, we define the index to be zero. From the above formulae it is obvious that the solution $x(t)$ will not contain derivatives of the function f if and only if $k \leq 1$. In that case the solution $x(t)$ is called *impulse free*.

Further, we assume that the degree of nilpotency of N is not more than one. Let $\begin{bmatrix} V & W \end{bmatrix}$ be an orthogonal matrix such that image V equals the image of E^T and image W equals the null space of E. Then $E = \begin{bmatrix} E_1 & 0 \end{bmatrix} \begin{bmatrix} V & W \end{bmatrix}^T = E_1 V^T$, where E_1 is full column rank. Since we assume the system has an index of at most one, we make the next assumptions (Engwerda and Salmah, 2009).

Assumption 1. : Throughout this paper the next assumptions are made w.r.t system (2.1) :

1. matrix E is singular
2. $\det(\lambda E - A) \neq 0$
3. $rank\left(\begin{bmatrix} E & AW \end{bmatrix}\right) = n + r$ (the system has index one, see (Kautsky, Nicholas and Chu, 1989)).

Finally, we define our main object of study in this paper, the open-loop Nash equilibrium (Engwerda and Salmah, 2009), (Engwerda, 2005).

Definition 2.1. Assume (2.1) is regular and has index one. Let x_0 be a consistent initial state and $\mathcal{U}$ denote the set of bounded piecewise continuous functions. Then $(u_1^*, u_2^*) \in \mathcal{U}$ is an open-loop Nash (OLN) equilibrium if for every $(u_1, u_2^*), (u_1^*, u_2) \in \mathcal{U}$, $J_1(u_1^*, u_2^*) \leq J_1(u_1, u_2^*)$ and $J_2(u_1^*, u_2^*) \leq J_2(u_1^*, u_2)$.

3 THE FINITE PLANNING HORIZON

In this section we consider the game (2.1) and (2.2) under the assumption that t_f is finite. Furthermore, to avoid the inclusion of controls in the scrap value, we make the standard assumption as in (Engwerda and Salmah, 2009) and (Mehrmann, Thoma and Wyner, 1991) that

$$X^T \bar{Q}_{it_f} X = \begin{bmatrix} Q_{it_f} & 0 \\ 0 & 0 \end{bmatrix}, \ i = 1, 2 \text{ , where } Q_{it_f} \in \mathbb{R}^{n \times n}.$$

It has been shown in (Engwerda and Salmah, 2009) that with the result from Theorem 2.1, the game (2.1) and (2.2) has a set of OLN equilibrium actions $(u_1(\cdot), u_2(\cdot))$ if and only if $(u_1(\cdot), u_2(\cdot))$ are OLN equilibrium actions for the game

$$\dot{x}_1(t) = Jx_1(t) + Y_1 B_1 u_1(t) + Y_1 B_2 u_2(t), \tag{3.1}$$
$$x_1(0) = \begin{bmatrix} I_n & 0 \end{bmatrix} X^{-1} x_0.$$

With cost functional J_i for the player i given by

$$\int_0^{t_f} \{z^T(t) M_i z(t)\} dt + x_1^T(t_f) Q_{it_f} x_1(t_f) \tag{3.2}$$

where $z^T(t) = \begin{bmatrix} x_1^T(t) & u_1^T(t) & u_2^T(t) \end{bmatrix}$ and

$$M_i = \begin{bmatrix} Q_i & V_i & W_i \\ V_i^T & R_{1i} & N_i \\ W_i^T & N_i^T & R_{2i} \end{bmatrix}. \tag{3.3}$$

The spellings of the matrices defined in (3.3) and another additional notation that will be used throughout this paper are presented in the Appendix. Next, we recall from (Engwerda and Salmah, 2009) the next result for nonzero-sum linear quadratic differential game for descriptor systems (see also (Engwerda, 2005) for the ordinary differential game).

Theorem 3.1. *A. Assume that $R_{ii} > 0$ and*
i. The Riccati differential equation

$$\dot{\tilde{P}}(t) = -\tilde{A}_2^T \tilde{P}(t) - \tilde{P}(t)\tilde{A} + \tilde{P}(t) B G^{-1} \tilde{B}^T \tilde{P}(t) - \tilde{Q}; \ \ \tilde{P}^T(t_f) = \begin{bmatrix} Q_{1t_f}^T & Q_{2t_f}^T \end{bmatrix} \tag{3.4}$$

has a solution $\tilde{P}$ on $[0, t_f]$ and
ii. The two Riccati differential equations

$$\begin{aligned} &\dot{K}_i(t) = -J^T K_i(t) - K_i(t) J + (K_i(t) Y_1 B_i + V_i) R_{ii}^{-1} \left(B_i^T Y_1^T K_i(t) + V_i^T\right) - Q_i; \\ &K_i(T) = Q_{it_f} \end{aligned} \tag{3.5}$$

have a symmetric solution $K_i(.)$ on $[0, t_f]$.
Then the differential game (2.1) and (2.2) has a unique OLN equilibrium for every consistent initial state. Moreover, the equilibrium actions are

$$\begin{bmatrix} u_1^*(t) \\ u_2^*(t) \end{bmatrix} = -G^{-1}\left(Z + \tilde{B}^T \tilde{P}(t)\right) \tilde{\Phi}(t, 0) \begin{bmatrix} I & 0 \end{bmatrix} X^{-1} x_0 \tag{3.6}$$

where $\Phi(t,0)$ *is the solution of the transition equation*

$$\dot{\tilde{\Phi}}(t,0) = \left(J - BG^{-1}\left(Z + \tilde{B}^T\tilde{P}(t)\right)\right)\tilde{\Phi}(t,0)\,; \;\; \tilde{\Phi}(0,0) = I.$$

The corresponding state trajectory is given by

$$x^*(t) = X\begin{bmatrix} x_1^*(t) \\ x_2^*(t) \end{bmatrix} \quad \textit{where} \quad x_1^*(t) = \tilde{\Phi}(t,0)\begin{bmatrix} I & 0 \end{bmatrix} X^{-1}x_0,$$

$$x_2^*(t) = Y_2\begin{bmatrix} B_1 & B_2 \end{bmatrix} G^{-1}\left(Z + \tilde{B}^T\tilde{P}(t)\right)x_1^*(t)\,.$$

B. For all $t_f \in [0,t_1)$ *there exist for all consistent* x_0 *a unique OLN equilibrium for game (2.1) and (2.2) if and only if the above Riccati differential equations i. and ii. have a solution for all* $t_f \in [0,t_1)$*.*

Using Theorem 3.1, we have the next necessary and sufficient conditions for the existence of an OLN equilibrium for open-loop zero-sum linear quadratic differential game on a finite planning horizon.

Theorem 3.2. *Consider the differential game described by (2.1), with for the player one, will minimize the quadratic cost function :*

$$J_1(u_1,u_2) = \int_0^{t_f}\left\{x^T(t)\bar{Q}x(t) + u_1^T(t)\bar{R}_1u_1(t) - u_2^T(t)\bar{R}_2u_2(t)\right\} + x^T(t_f)Q_{t_f}x(t_f) \tag{3.7}$$

$$= \int_0^{t_f}\left\{z^T(t)\bar{M}z(t)\right\}dt + x_1^T(t_f)Q_{t_f}x_1(t_f)$$

where

$$\bar{M} = \begin{bmatrix} Q & V & W \\ V^T & R_{\bar{1}\bar{1}} & N \\ W^T & N^T & R_{\bar{2}\bar{2}} \end{bmatrix}.$$

And, for player two, will minimize the opposite objective function

$$J_2(u_1,u_2) = -J_1(u_1,u_2)\,,$$

where the matrices $\bar{Q}, \bar{Q}_{t_f}$ *and* $\bar{R}_i,\ i=1,2$ *are symmetric. Moreover, assume that* $\bar{R}_i, R_{\bar{i}\bar{i}},\ i = 1,2$ *are positive definite. Then, for all* $t_f \in [0,t_1)$*, this zero-sum linear quadratic differential game has for every initial state an OLN equilibrium if and only if the following two conditions hold on* $[0,t_1)$*.*

1. The Riccati differential equation

$$\dot{P}(t) = -\tilde{J}^TP(t) - P(t)\tilde{J} + P(t)B\hat{G}^{-1}B^TP(t) - \hat{Q},\;\; P(t_f) = Q_{t_f} \tag{3.8}$$

has a symmetric solution $P(0,t_f)$ *for all* $t_f \in [0,t_1)$*.*

2. The Riccati differential equations

$$\begin{aligned} &\dot{K}_1(t) = -J^TK_1(t) - K_1(t)J + (K_1(t)Y_1B_1 + V_1)R_{11}^{-1}\left(B_1^TY_1^TK_1(t) + V_1^T\right) - Q; \\ &K_1(T) = Q_{t_f} \end{aligned} \tag{3.9}$$

$$\begin{array}{l}\dot{K}_2(t) = -J^T K_2(t) - K_2(t) J + (K_2(t) Y_1 B_2 + V_2) R_{22}^{-1} \left(B_2^T Y_1^T K_2(t) + V_2^T\right) + Q; \\ K_2(T) = -Q_{t_f}\end{array} \tag{3.10}$$

have a solution $K_i(.)$ *on* $[0, t_f]$, $i = 1, 2$. *Moreover, if the above conditions are satisfied the equilibrium is unique. In that case the equilibrium actions are*

$$\begin{bmatrix} u_1^*(t) \\ u_2^*(t) \end{bmatrix} = -\bar{I}\bar{G}^{-1}\left(\bar{Z} + \tilde{B}^T \bar{P}(t)\right)\tilde{\Phi}(t,0)\begin{bmatrix} I & 0 \end{bmatrix} X^{-1}x_0. \tag{3.11}$$

Here $\tilde{\Phi}(t, 0)$ *satisfies the transition equation*

$$\dot{\tilde{\Phi}}(t,0) = \left(J - B\bar{G}^{-1}\left(\bar{Z} + \tilde{B}^T \bar{P}(t)\right)\right)\tilde{\Phi}(t,0); \;\; \tilde{\Phi}(0,0) = I.$$

Proof. First notice that $G^{-1}Z = \hat{G}^{-1}Z_1$ and $\tilde{Q}_2 = -\tilde{Q}_1$. Premultiplication of (3.4) with the matrix $\begin{bmatrix} I & I \end{bmatrix}$ gives then the next differential equation in $P(t) := (P_1 + P_2)(t)$

$$\begin{array}{l}\dot{P}(t) = J^T P(t) - P(t)\left(J - B\bar{G}^{-1}Z_1\right) + P(t) B G^{-1}\begin{bmatrix} B_1^T Y_1^T P_1(t) \\ B_2^T Y_1^T P_2(t) \end{bmatrix}, \\ P(t_f) = 0.\end{array}$$

Obviously $P(t) = 0$ satisfies this differential equation. Since the solution to this differential equation is unique we conclude that $P_1(t) = -P_2(t)$. Substitution of this into equation (3.4) again, yields:

$$\begin{array}{l}\dot{P}_1(t) = -J^T P_1 - \left(Z_1 G^{-1}\tilde{B}^T \begin{bmatrix} I \\ I \end{bmatrix}\right) P_1 - P_1\tilde{A} + P_1 B\hat{G}^{-1}\bar{I}B^T P_1 - \left(Q_1 - Z_1\hat{G}^{-1}Z_1\right), \\ P_1(t_f) = Q_{t_f}.\end{array}$$

After some elementary manipulations we get then the differential equation (3.8) with $P(t) =: P_1(t)$. The symmetry of $P(0, t_f)$ follows from the symmetry of Q_{t_f} and the symmetry of the differential equation (3.8) (Engwerda, 2005). The corresponding equilibrium strategies are then directly obtained from Theorem 3.1. □

4 THE INFINITE PLANNING HORIZON

As in (Engwerda and Salmah, 2009), in this section we assume that the cost functional player $i = 1, 2$, likes to be minimize is

$$\lim_{t_f \to \infty} J_i(x_0, u_1, u_2, t_f), \tag{4.1}$$

where

$$J_i(x_0, u_1, u_2, t_f) = \int_0^{t_f} \{x^T(t)\bar{Q}_i x(t) + u_i^T(t)\bar{R}_i u_i(t)\} dt$$

subject to (2.1).

We start by recalling some important notions concerning algebraic Riccati equations that play a main role in the rest of this paper ((Engwerda and Salmah, 2009), (Engwerda, 2005)).

Definition 4.1. A solution $P \in \mathbb{R}^{2n\times n}$ of the algebraic Riccati equation

$$0 = \tilde{A}_2^T\tilde{P} + \tilde{P}\tilde{A} - \tilde{P}BG^{-1}\tilde{B}^T\tilde{P} + \tilde{Q} \tag{4.2}$$

is called

a. stabilizing, if $\sigma\left(\tilde{A} - BG^{-1}\tilde{B}^T\tilde{P}\right) \subset \mathbb{C}^-$, where $\sigma(A)$ denote the spectrum of matrix A and $\mathbb{C}^- = \{\lambda \in \mathbb{C}\,|\mathrm{Re}(\lambda) < 0\}$.

b. left-right stabilizing (LRS) if

1. it is a stabilizing solution, and
2. $\sigma\left(-\tilde{A}_2^T + \tilde{P}BG^{-1}\tilde{B}^T\right) \subset \mathbb{C}_0^+$ where $\mathbb{C}_0^+ = \{\lambda \in \mathbb{C}\,|\mathrm{Re}(\lambda) \geq 0\}$.

The following theorem states the existence of an OLN equilibrium for the infinite planning horizon open-loop nonzero-sum linear quadratic differential game (Engwerda and Salmah, 2009).

Theorem 4.1. *Assume that $R_{ii} > 0$ and*

1. The algebraic Riccati equation (4.2) has a LRS solution, and

2. The two algebraic Riccati equations

$$0 = J^TK_i + K_iJ - (K_iY_1B_i + V_i)R_{ii}^{-1}\left(B_i^TY_1^TK_i + V_i^T\right) + Q_i, \quad i = 1,2 \tag{4.3}$$

have a stabilizing solution $K_i,\ i = 1,2$.

Then the linear quadratic differential game (2.1) and (4.1) has an OLN equilibrium for every consistent initial state.

Moreover, one set of equilibrium actions is (for $t > 0$) given by :

$$\begin{bmatrix} u_1^*(t) \\ u_2^*(t) \end{bmatrix} = -G^{-1}\left(Z + \tilde{B}^T\tilde{P}\right)\tilde{\Phi}(t,0)\begin{bmatrix} I & 0 \end{bmatrix}X^{-1}x_0 \tag{4.4}$$

where $\tilde{\Phi}(t,0)$ is the solution of the transition equation

$$\dot{\tilde{\Phi}}(t,0) = \left(J - BG^{-1}\left(Z + \tilde{B}^T\tilde{P}\right)\right)\tilde{\Phi}(t,0)\,; \;\; \tilde{\Phi}(0,0) = I.$$

The corresponding state trajectory is given by

$$x^*(t) = X\begin{bmatrix} x_1^*(t) \\ x_2^*(t) \end{bmatrix} \;\; \textit{where} \;\; x_1^*(t) = \tilde{\Phi}(t,0)\begin{bmatrix} I & 0 \end{bmatrix}X^{-1}x_0,$$

$$x_2^*(t) = Y_2\begin{bmatrix} B_1 & B_2 \end{bmatrix}G^{-1}\left(Z + \tilde{B}^T\tilde{P}\right)x_1^*(t)\,.$$

Furthermore, the costs by using the actions (4.4) for the players are

$$\left(\begin{bmatrix} I & 0 \end{bmatrix}X^{-1}x_0\right)^T\bar{L}_i\begin{bmatrix} I & 0 \end{bmatrix}X^{-1}x_0, \;\; i = 1,2,$$

where, with $A_{cl} := J - BG^{-1}\tilde{B}^T\tilde{P}$, $\bar{L}_i$ is the unique solution of the Lyapunov equation

$$\begin{bmatrix} I & \left(-G^{-1}\left(Z + \tilde{B}^T\tilde{P}\right)\right)^T \end{bmatrix}\bar{L}_i\begin{bmatrix} I & \left(-G^{-1}\left(Z + \tilde{B}^T\tilde{P}\right)\right)^T \end{bmatrix}^T + A_{cl}^T\bar{L}_i + \bar{L}_iA_{cl} = 0. \tag{4.5}$$

A direct corollary from Theorem 3.2 and Theorem 4.1 is our next Theorem 4.2 about the infinite planning horizon open-loop zero-sum linear quadratic differential game.

Theorem 4.2. *Consider the differential game described by (2.1) with, for player one, will minimize the quadratic cost functional :*

$$J_1(u_1,u_2) = \int_0^\infty \left\{x^T(t)\bar{Q}x(t) + u_1^T(t)\bar{R}_1u_1(t) - u_2^T(t)\bar{R}_2u_2(t)\right\}dt \tag{4.6}$$

$$= \int_0^\infty \left\{z^T(t)\bar{M}z(t)\right\}dt$$

where

$$\bar{M} = \begin{bmatrix} Q & V & W \\ V^T & R_{\bar{1}\bar{1}} & N \\ W^T & N^T & R_{\bar{2}\bar{2}} \end{bmatrix}.$$

And, for player two, will minimize the opposite objective function

$$J_2(u_1,u_2) = -J_1(u_1,u_2),$$

where the matrices $\bar{Q}$ and $\bar{R}_i,\ i=1,2$ are symmetric. Moreover, assume that $\bar{R}_i, R_{\bar{i}\bar{i}},\ i=1,2$ are positive definite. Then this infinite planning horizon open-loop zero-sum linear quadratic differential game has

1. An OLN equilibrium for every initial state if and only if the following two conditions hold

1. *the coupled algebraic Riccati equations*

$$0 = \tilde{J}^TP_1 + P_1\tilde{J} - P_1B\hat{G}^{-1}B^TP_1 + \hat{Q} \tag{4.7}$$

$$0 = \tilde{J}^TP_2 + P_2\tilde{J} + P_2B\hat{G}^{-1}B^TP_2 - \hat{Q} \tag{4.8}$$

have a set of solution $\tilde{P}$ such that $J - B\bar{G}^{-1}\left(\bar{Z} + \tilde{B}^T\tilde{P}\right)$ is stable; and

2. *the two algebraic Riccati equations*

$$0 = J^TK_1 + K_1J - (K_1Y_1B_1 + V_1)R_{11}^{-1}\left(B_1^TY_1^TK_1 + V_1^T\right) + Q \tag{4.9}$$

$$0 = J^TK_2 + K_2J - (K_2Y_1B_2 + V_2)R_{22}^{-1}\left(B_2^TY_1^TK_2 + V_2^T\right) - Q \tag{4.10}$$

have a symmetric solution K_i such that $J - Y_1B_iR_{ii}^{-1}\left(V_i^T + B_i^TY_1^TK_i\right)$ is stable, $i=1,2.$ Moreover, the corresponding equilibrium actions are

$$\begin{bmatrix} u_1^*(t) \\ u_2^*(t) \end{bmatrix} = -\bar{G}^{-1}\left(\bar{Z} + \tilde{B}^T\tilde{P}\right)x_1(t) \tag{4.11}$$

where $x_1(t)$ satisfies the differential equation

$$\dot{x_1}(t) = \left(J - B\bar{G}^{-1}\left(\bar{Z} + \tilde{B}^T\tilde{P}\right)\right)x_1(t),\quad x_1(0) = \begin{bmatrix} I_n & 0 \end{bmatrix}X^{-1}x_0.$$

2. A unique OLN equilibrium for every initial state if and only if the set of coupled algebraic Riccati equations (4.7) and (4.8) has a LRS solution and the two algebraic Riccati equations (4.9) and (4.10) have a symmetric stabilizing solution. The corresponding equilibrium actions are as described in item 1.

In the case that matrix J is stable, then the condition under which the zero-sum game has an OLN equilibrium can be further simplified. This is shown in the following theorem, while the proof of this theorem is in line with the proof in (Engwerda, 2005).

Theorem 4.3. *Consider the open-loop zero-sum differential game as described in Theorem 4.2. Assume, additionally, that matrix J is stable. Then, the game has for every initial state a unique OLN equilibrium if and only if the next two conditions hold.*
1. The algebraic Riccati equation

$$0 = -\tilde{J}^T P - P\tilde{J} + PB\hat{G}^{-1}B^T P - \hat{Q}, \tag{4.12}$$

has a symmetric solution P such that $J - B\bar{G}^{-1}\left(\bar{Z} + \tilde{B}^T\bar{P}\right)$ is stable.
2. The two algebraic Riccati equations (4.9) and (4.10) have a symmetric solution K_i such that $J - Y_1 B_i R_{ii}^{-1}\left(V_i^T + B_i^T Y_1^T K_i\right)$ is stable, $i = 1, 2$. Moreover, the corresponding unique equilibrium actions are

$$\begin{bmatrix} u_1^*(t) \\ u_2^*(t) \end{bmatrix} = -\bar{I}\bar{G}^{-1}\left(\bar{Z} + \tilde{B}^T\bar{P}\right)x_1(t) \tag{4.13}$$

where $x_1(t)$ satisfies the differential equation

$$\dot{x_1}(t) = \left(J - B\bar{G}^{-1}\left(\bar{Z} + \tilde{B}^T\bar{P}\right)\right)x_1(t), \quad x_1(0) = \begin{bmatrix} I_n & 0 \end{bmatrix} X^{-1}x_0.$$

The cost for player one is

$$J_1 = \left(\begin{bmatrix} I & 0 \end{bmatrix} X^{-1}x_0\right)^T P \begin{bmatrix} I & 0 \end{bmatrix} X^{-1}x_0$$

and for player two is $-J_1$.

Proof. According to Theorem 4.2 this game has a unique OLN equilibrium for every initial state if and only if the two coupled algebraic Riccati equations (4.7) and (4.8) have a LRS solution $P_i,\ i = 1, 2$ and the two algebraic Riccati equations (4.9) and (4.10) have a stabilizing solution. Adding equation (4.7) to (4.8) yields the following differential equation in $(P_1 + P_2)$

$$0 = -\tilde{J}^T(P_1 + P_2) + (P_1 + P_2)\left(\tilde{J} - B\hat{G}^{-1}B^T P_1 - B\hat{G}^{-1}B^T P_2\right).$$

This is a Lyapunov equation of the form $A^T X + XC^T = 0$, with $X = P_1 + P_2$ and $C^T = \tilde{J} - B\hat{G}^{-1}B^T P_1 - B\hat{G}^{-1}B^T P_2$. Now, independent of the specification of (P_1, P_2), C is always stable. So, whatever $P_i,\ i = 1, 2$ are, this Lyapunov equation has a unique solution $(P_1 + P_2)$. Obviously $P_1 + P_2 = 0$ satisfies this Lyapunov equation. So, necessary we conclude that $P_1 = -P_2$. Substitution of this into equation (4.7) then shows that equations (4.7) and (4.8) have a solution P_i if and only if equation (4.12) has a solution P. The corresponding equilibrium strategies then follow directly from Theorem 4.2. The symmetric and uniqueness properties of P follow immediately from the fact that P is stabilizing solution of an ordinary Riccati equation (see (Engwerda, 2005)). $\square$

In case it is additionally assumed in Theorem 4.3 that Q is positive semidefinite, equation (4.12) has a stabilizing solution (see (Engwerda, 2005) for the ordinary differential game). Moreover in that case, by considering the fact that equation (4.10) should have a stabilizing solution it can be rephrased as that the following Riccati equation should have a solution K such that $J + Y_1 B_2 R_{22}^{-1} \left(V_2^T + B_2^T Y_1^T K \right)$ is stable :

$$0 = J^T K + KJ - \left(K Y_1 B_2 + V_2 \right) R_{22}^{-1} \left(B_2^T Y_1^T K + V_2^T \right) - Q. \tag{4.14}$$

We present this consequence in the next theorem.

Theorem 4.4. *Consider the open-loop zero-sum differential game as described in Theorem 4.2. Assume that matrix J is stable and $Q \geq 0$. Then this game has, for every initial state, a unique OLN equilibrium if and only if the algebraic Riccati equations*

$$0 = -\tilde{J}^T P - P\tilde{J} + PB\hat{G}^{-1} B^T P - \hat{Q} \tag{4.15}$$

and

$$0 = J^T K + KJ - \left(K Y_1 B_2 + V_2 \right) R_{22}^{-1} \left(B_2^T Y_1^T K + V_2^T \right) - Q \tag{4.16}$$

have a solution P and K, respectively, such that $J - B\bar{G}^{-1} \left(\bar{Z} + \tilde{B}^T \bar{P} \right)$ and $J + Y_1 B_2 R_{22}^{-1} \left(V_2^T + B_2^T Y_1^T K \right)$ are stable. Moreover, the corresponding unique equilibrium actions are

$$\begin{bmatrix} u_1^* (t) \\ u_2^* (t) \end{bmatrix} = -\bar{I}\bar{G}^{-1} \left(\bar{Z} + \tilde{B}^T \bar{P} \right) x_1 (t) .$$

Here $x_1 (t)$ satisfies the differential equation

$$\dot{x}_1 (t) = \left(J - B\bar{G}^{-1} \left(\bar{Z} + \tilde{B}^T \bar{P} \right) \right) x_1 (t), \quad x_1 (0) = \begin{bmatrix} I_n & 0 \end{bmatrix} X^{-1} x_0.$$

The cost for player one is

$$J_1 = \left(\begin{bmatrix} I & 0 \end{bmatrix} X^{-1} x_0 \right)^T P \begin{bmatrix} I & 0 \end{bmatrix} X^{-1} x_0$$

and for player two is $-J_1$.

5 INVARIANT SUBSPACE METHOD

In this section we study the relationship between certain invariant subspaces of matrix $\bar{M}$ and solutions of the algebraic Riccati equation (4.7) and (4.8). We start by stating two basic lemmas (Engwerda, 2005).

Lemma 5.1. *Let $C \in \mathbb{R}^{3n}$ be an n-dimensional invariant subspaces of $\bar{M}$, and let $C_i \in \mathbb{R}^{n \times n}$, $i = 0, 1, 2$, be three real matrices such that $C = \operatorname{Im} \begin{bmatrix} C_0^T & C_1^T & C_2^T \end{bmatrix}^T$. If C_0 is invertible then $P_i := C_i C_0^{-1}$, $i = 1, 2$, solves (4.7) and (4.8) and $J - B\bar{G}^{-1} \left(\bar{Z} + \tilde{B}^T \tilde{P} \right) = \sigma \left(\bar{M} \mid_C \right)$. Furthermore, (P_1, P_2) is independent of the specific choice of basis of C.*

A subspace C that satisfies the above property is called a graph subspace.

Lemma 5.2. *Let $P_i \in \mathbb{R}^{n\times n}$, $i=1,2$, be a solution to the set of coupled Riccati equation (4.7) and (4.8). Then there exist matrices $C_i \in \mathbb{R}^{n\times n}$, $i=0,1,2$, with C_0 is invertible, such that $P_i = C_i C_0^{-1}$, $i=1,2$. Furthermore, the columns of $\left[\begin{array}{ccc} C_0^T & C_1^T & C_2^T \end{array}\right]^T$ form a basis of n-dimensional invariant subspaces of $\bar{M}$.*

Now, we introduce a separate notation for the set of $\bar{M}$-invariant subspaces

$$\bar{\mathcal{M}}^{inv} := \{\mathcal{T}\,|\bar{M}\mathcal{T} \subset \mathcal{T}\}.$$

It follows from above Lemma 5.1 and Lemma 5.2 that the following set of graph subspaces plays a crucial role

$$\mathcal{P}^{pos} := \left\{ \mathcal{P} \in \bar{\mathcal{M}}^{inv} \,\middle|\, \mathcal{P} \oplus \mathrm{Im}\begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix} = \mathbb{R}^{3n} \right\}.$$

Every element of $\mathcal{P}^{pos}$ defines exactly one solution of the algebraic Riccati equation. The following fact relates solutions of algebraic Riccati equation (4.7) and (4.8) with $\bar{M}$-invariant subspaces in $\mathcal{P}^{pos}$ that follows immediately from Lemma 5.1 and Lemma 5.2.

Corollary 5.3. *Equation (4.7) and (4.8) have a set of stabilizing solution (P_1, P_2) if and only if there exists an $\bar{M}$-invariant subspace $\mathcal{P}$ in $\mathcal{P}^{pos}$ such that $\mathrm{Re}\lambda < 0$ for all $\lambda \in \sigma\left(\bar{M}\,|_{\mathcal{P}}\right)$.*

In general, the set of algebraic Riccati equations (4.7) and (4.8) does not have a unique stabilizing solution. The following proposition shows, however, that it does have a unique LRS solution.

Proposition 5.4. *1. The set of algebraic Riccati equations (4.7) and (4.8) has a LRS solution (P_1, P_2) if and only if matrix $\bar{M}$ has an n-dimensional stable graph subspace and $\bar{M}$ has $2n$ eigenvalues (counting algebraic multiplicities) in $\mathbb{C}_0^+$.*
2. If the set of algebraic Riccati equations (4.7) and (4.8) has a LRS solution, then it is unique.

Now, we will use the theory of invariant subspace above to find OLN equilibria of the infinite planning horizon open-loop zero-sum differential game. The proof of the next theorems and corollaries are in line with the proof in (Engwerda, 2005). We start with the following theorem.

Theorem 5.5. *If the open-loop zero-sum differential game (2.1) and (4.6) has an OLN equilibrium for every consistent initial state, then*

1. *$\bar{M}$ has at least n stable eigenvalues (counted with algebraic multiplicities). More in particular, there exist a p-dimensional stable $\bar{M}$-invariant subspace S, with $p \geq n$, such that $\mathrm{Im}\left[\begin{array}{ccc} I & C_1^T & C_2^T \end{array}\right]^T \subset S$ for some $C_i \in \mathbb{R}^{n\times n}$, $i=1,2$.*
2. *the two algebraic Riccati equations (4.9) and (4.10) have a stabilizing solution.*

Conversely, if the two algebraic Riccati equation (4.9) and (4.10) have a stabilizing solution and $v^T(t) =: \left[\begin{array}{ccc} x_1^T(t) & \psi_1^T(t) & \psi_2^T(t) \end{array}\right]$ is an asymptotically stable solution of

$$\dot{v}(t) = \bar{M}v(t), \quad x_1(0) = \left[\begin{array}{cc} I & 0 \end{array}\right] X^{-1} x_0$$

then, with $\psi^T = \left[\begin{array}{cc} \psi_1^T(t) & \psi_2^T(t) \end{array}\right]$, $\left[\begin{array}{c} u_1^*(t) \\ u_2^*(t) \end{array}\right] = -\bar{G}^{-1}\left(\tilde{B}^T\psi(t) + \bar{Z}x_1(t)\right)$ *provides an OLN equilibrium for the open-loop zero-sum linear quadratic differential game (2.1) and (4.6). The corresponding state trajectory is given by*

$$x^*(t) = X\left[\begin{array}{c} x_1^*(t) \\ x_2^*(t) \end{array}\right], \quad \text{where} \quad x_2^*(t) = Y_2\left[\begin{array}{cc} B_1 & B_2 \end{array}\right]\bar{G}^{-1}\left(\tilde{B}^T\psi(t) + \bar{Z}x_1^*(t)\right).$$

Combining Lemma 5.1 and Theorem 4.2 the following fact (Engwerda and Salmah, 2009) results.

Corollary 5.6. *An immediate consequence of Lemma 5.1 and Theorem 4.2 is that if* $\bar{M}$ *has a stable invariant graph subspace and the two algebraic Riccati equation (4.9) and (4.10) have a stabilizing solution, the game will have at least one OLN set of equilibrium actions.*

If the equilibrium actions allow for a feedback synthesis then the closed-loop dynamic of the game can be described by (Engwerda and Salmah, 2009)

$$\dot{x}_1(t) = \left(J - B\bar{G}^{-1}\left(\bar{Z} + \tilde{B}^T\tilde{P}\right)\right)x_1(t), \quad x_1(0) = \left[\begin{array}{cc} I_n & 0 \end{array}\right]X^{-1}x_0. \tag{5.1}$$

In this case the equilibrium actions are given by

$$\left[\begin{array}{c} u_1^*(t) \\ u_2^*(t) \end{array}\right] = -\bar{G}^{-1}\left(\bar{Z} + \tilde{B}^T\tilde{P}\right)x_1(t). \tag{5.2}$$

For the case above, the following theorem holds.

Theorem 5.7. *Assume the open-loop zero-sum linear quadratic differential game (2.1) and (4.6) has an OLN equilibrium for every initial state and the equilibrium control actions allow for a feedback synthesis. Then the following statements are true.*

1. $\bar{M}$ *has at least* n *stable eigenvalues (counted with algebraic multiplicities). In particular, for each such OLN equilibrium there exits a uniquely determined* n*-dimensional stable* $\bar{M}$*-invariant subspace* $\mathrm{Im}\left[\begin{array}{ccc} I & V_1^T & V_2^T \end{array}\right]^T$ *for some* $V_i \in \mathbb{R}^{n\times n}$.
2. *The following algebraic Riccati equation (4.9) and (4.10) have a symmetric solution* K_i *such that* $J - Y_1B_iR_{ii}^{-1}\left(V_i^T + B_i^TY_1^TK_i\right)$ *is stable,* $i = 1, 2$.

Then combining the results of both Theorem 5.5 and Theorem 5.7 yields the following.

Corollary 5.8. *The open-loop zero-sum differential game (2.1) and (4.6) has, for every initial state, an OLN equilibrium* (u_1^*, u_2^*) *if and only if*

1. *there exist* P_1 *and* P_2 *which are solutions of the set of coupled algebraic Riccati equation (4.7) and (4.8) satisfying the additional constraint that the eigenvalues of* $\bar{A}_{cl} := J - B\bar{G}^{-1}\left(\bar{Z} + \tilde{B}^T\tilde{P}\right)$ *are all situated in the left-half complex plane, and*
2. *the two algebraic Riccati equation (4.9) and (4.10) have a symmetric solution* K_i *such that* $J - Y_1B_iR_{ii}^{-1}\left(V_i^T + B_i^TY_1^TK_i\right)$ *is stable,* $i = 1, 2$.

If (P_1, P_2) is a set of stabilizing solutions of the coupled algebraic Riccati equation (4.7) and (4.8), the actions

$$\begin{bmatrix} u_1^*(t) \\ u_2^*(t) \end{bmatrix} = -\bar{G}^{-1}\left(\bar{Z} + \tilde{B}^T\tilde{P}\right)\Phi(t,0)\,x_0 \tag{5.3}$$

where $\Phi(t,0)$ satisfies the transition equation $\dot{\Phi}(t,0) = \bar{A}_{cl}\Phi(t,0)\,; \;\; \Phi(0,0) = I$, yield an OLN equilibrium.

The condition under which the game has an unique OLN equilibrium for every initial state is given in the following theorem.

Theorem 5.9. *Consider the open-loop zero-sum differential game (2.1) and (4.6). This game has a unique OLN equilibrium for every consistent initial state if and only if*

1. *equation (4.7) and (4.8) have a LRS solution, and*
2. *the two algebraic Riccati equation (4.9) and (4.10) have a stabilizing solution.*

Moreover, the unique equilibrium actions are given by (4.11).

6 AN EXAMPLE

Consider the game defined by the system

$$E\dot{x}(t) = Ax(t) + B_1u_1(t) + B_2u_2(t)\,, \quad x(0) = x_0$$

and cost function

$$J_1(u_1, u_2) = \int_0^\infty \left\{x^T(t)\,\bar{Q}x(t) + u_1^T(t)\,\bar{R}_1u_1(t) - u_2^T(t)\,\bar{R}_2u_2(t)\right\}dt$$

where $E = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$, $B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\bar{Q} = \begin{bmatrix} 5 & 1 \\ 0 & 1 \end{bmatrix}$, $\bar{R}_1 = [1]$, and $\bar{R}_2 = [3]$.
With $Y^T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ the matrix pencil (E, A) can be rewritten into its Weierstrass canonical form (2.5) where $X_1 = Y_2^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $X_2 = Y_1^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $N = [0]$, and $J = [1]$. Furthermore, after some calculations, matrix $\bar{M}$ results as

$$\bar{M} = \begin{bmatrix} 5 & -6 & -6 \\ -6 & 8 & 7 \\ -6 & 7 & 4 \end{bmatrix}.$$

The egenvalues of $\bar{M}$ are $-1.8972, 0.3698,$ and 18.5274. The corresponding eigenvectors are

$$\begin{bmatrix} -0.4497 \\ 0.3177 \\ -0.8347 \end{bmatrix}, \begin{bmatrix} 0.7194 \\ 0.6828 \\ -0.1277 \end{bmatrix}, \text{ and } \begin{bmatrix} -0.5294 \\ 0.6579 \\ 0.5357 \end{bmatrix} \text{ respectively.}$$

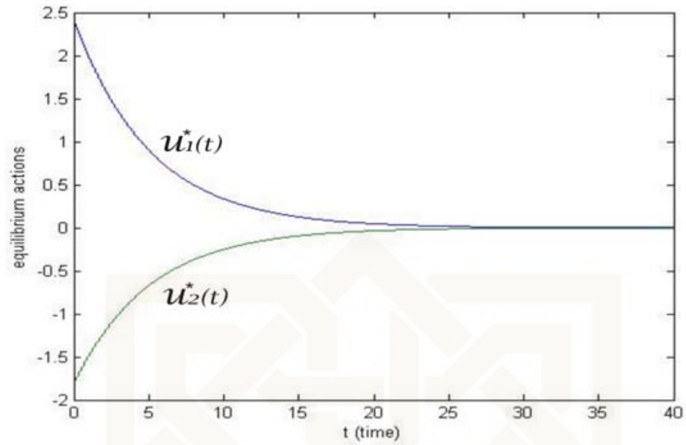

Figure 1: equilibrium actions (u_1^*, u_2^*)

From this we observe that $\bar{M}$ has one stable eigenvalue and $\mathcal{P}^{pos}$ has one element that is the eigenspace

$$\mathrm{Im}\begin{bmatrix} -0.4497 \\ 0.3177 \\ -0.8347 \end{bmatrix}$$

corresponding to the eigenvalue -1.8972. According to Lemma 5.1 and Lemma 5.2 , then

$$P_1 = 0.3177 \cdot -0.4497^{-1} = -0.7065$$
$$P_2 = -0.8347 \cdot -0.4497^{-1} = 1.8561$$

provide a stabilizing solution of the set of coupled algebraic Riccati equations. Using (4.11), we obtain the equilibrium actions for the players are

$$\begin{bmatrix} u_1^*(t) \\ u_2^*(t) \end{bmatrix} = \begin{bmatrix} 2.4212 \\ -1.8090 \end{bmatrix} e^{(-0.1968)t}.$$

Because $\bar{M}$ has one stable eigenvalue, then according to Proposition 5.4 and Theorem 5.9 the game has unique OLN equilibrium. Figure 1 illustrates the equilibrium actions of the game.

7 CONCLUDING REMARKS

This paper studies the linear quadratic zero-sum differential game for descriptor systems which have index one. Necessary and sufficient conditions for the existence of an OLN equilibrium have been derived. The paper shows how the solution of the game depends on a Riccati differential equation for the finite horizon case and an algebraic Riccati equation for the infinite horizon. A numerical example illustrating some of the theoretical results is presented.

The problem addressed in this paper is restricted to index one descriptor systems. To find an OLN equilibrium in a zero-sum game that has higher order index is still an open problem to be analyzed.

APPENDIX

We use the next shorthand notation in this paper :

$$Q_i := X_1^T \bar{Q}_i X_I, \quad N_i := B_1^T Y_2^T X_2^T \bar{Q}_i X_2 Y_2 B_2, \quad V_i := -X_1^T \bar{Q}_i X_2 Y_2 B_1, \quad W_i := -X_1^T \bar{Q}_i X_2 Y_2 B_2,$$
$$Q := X_1^T \bar{Q} X_I, \quad N := B_1^T Y_2^T X_2^T \bar{Q} X_2 Y_2 B_2, \quad V := -X_1^T \bar{Q} X_2 Y_2 B_1, \quad W := -X_1^T \bar{Q} X_2 Y_2 B_2,$$
$$R_{11} := B_1^T Y_2^T X_2^T \bar{Q}_1 X_2 Y_2 B_1 + \bar{R}_1, \qquad R_{12} := B_1^T Y_2^T X_2^T \bar{Q}_2 X_2 Y_2 B_1,$$
$$R_{21} := B_2^T Y_2^T X_2^T \bar{Q}_1 X_2 Y_2 B_2, \qquad R_{22} := B_2^T Y_2^T X_2^T \bar{Q}_2 X_2 Y_2 B_2 + \bar{R}_2,$$
$$R_{\bar{1}\bar{1}} := B_1^T Y_2^T X_2^T \bar{Q} X_2 Y_2 B_1 + \bar{R}_1, \qquad R_{\bar{2}\bar{2}} := B_2^T Y_2^T X_2^T \bar{Q} X_2 Y_2 B_2 - \bar{R}_2,$$
$$A_2 = \mathrm{diag}\{J, J\}, \quad B = Y_1 \begin{bmatrix} B_1 & B_2 \end{bmatrix}, \ \tilde{B}^T = \mathrm{diag}\{B_1^T Y_1^T, B_2^T Y_1^T\}, \quad \tilde{A} = J - BG^{-1}Z,$$
$$Z_i = \begin{bmatrix} V_i & W_i \end{bmatrix}, \ \tilde{Q}_i = Q_i - Z_i G^{-1} Z, \ \tilde{Z} = \begin{bmatrix} V & W \end{bmatrix}, \ \hat{Q} = Q + \tilde{Z}\bar{G}^{-1}\bar{Z}, \ \tilde{J} = J - B\bar{G}^{-1}\bar{Z}.$$
$$\tilde{P} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}, \ \tilde{A}_2^T = A_2^T - \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} G^{-1}\tilde{B}^T, \ \tilde{Q} = \begin{bmatrix} \tilde{Q}_1 \\ \tilde{Q}_2 \end{bmatrix}, \ Z = \begin{bmatrix} V_1^T \\ W_2^T \end{bmatrix}, \ G = \begin{bmatrix} R_{11} & N_1 \\ N_2^T & R_{22} \end{bmatrix},$$
$$\bar{Z} = \begin{bmatrix} V^T \\ -W^T \end{bmatrix}, \ \bar{G} = \begin{bmatrix} R_{\bar{1}\bar{1}} & N \\ -N^T & -R_{\bar{2}\bar{2}} \end{bmatrix}, \ \hat{G} = \begin{bmatrix} R_{\bar{1}\bar{1}} & N \\ N^T & R_{\bar{2}\bar{2}} \end{bmatrix}, \ \bar{P} = \begin{bmatrix} P \\ P \end{bmatrix}, \bar{B}^T = \begin{bmatrix} B_1^T Y_1^T \\ -B_2^T Y_1^T \end{bmatrix},$$
$$\bar{I} = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.$$

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